

The Surface Laplacian Technique in EEG: Theory and Methods

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Abstract

This paper reviews the method of surface Laplacian differentiation to study EEG. We focus on topics that are helpful for a clear understanding of the underlying concepts and its efficient implementation, which is especially important for EEG researchers unfamiliar with the technique. The popular methods of finite difference and splines are reviewed in detail. The former has the advantage of simplicity and low computational cost, but its estimates are prone to a variety of errors due to discretization. The latter eliminates all issues related to discretization and incorporates a regularization mechanism to reduce spatial noise, but at the cost of increasing mathematical and computational complexity. These and several other issues deserving further development are highlighted, some of which we address to the extent possible. Here we develop a set of discrete approximations for Laplacian estimates at peripheral electrodes. We also provide the mathematical details of finite

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difference approximations that are missing in the literature, and discuss the problem of computational performance, which is particularly important in the context of EEG splines where data sets can be very large. Along this line, the matrix representation of the surface Laplacian operator is carefully discussed and some figures are given illustrating the advantages of this approach. In the final remarks, we briefly sketch a possible way to incorporate finite-size electrodes into Laplacian estimates that could guide further developments.

Key words: surface Laplacian, surface Laplacian matrix, high-resolution EEG, EEG regularization, spline Laplacian, discrete Laplacian

1. Introduction

The surface Laplacian technique is a powerful method to study EEG. It emerged from the seminal works of Nicholson [1973], Freeman and Nicholson [1975], Nicholson and Freeman [1975], and Hjorth [1975]. These were followed by efforts to develop better computational methods [see Gevins, 1988, 1989, Gevins et al., 1990, Perrin et al., 1989, Law et al., 1993a, Yao, 2002, Carvalhaes and Suppes, 2011] as well as attempts to combine the surface Laplacian with other methods [Kayser and Tenke, 2006b,a, Carvalhaes et al., 2009], making the technique increasingly popular among EEG researchers. For instance, modern applications include studies on generators of event-related potentials [Kayser and Tenke, 2006b,a], quantitative EEG [Tenke et al., 2011], spectral coherence [Srinivasan et al., 2007, Winter et al., 2007], event-related

synchronization/desynchronization [Del Percio et al., 2007], phase-lock synchronization [Doesburg et al., 2008], estimation of cortical connectivity [Astolfi et al., 2007], high-frequency EEG [Fitzgibbon et al., 2013], and brain-computer interface [Lu et al., 2013], just to mention a few.

The motivation for using the surface Laplacian is grounded on Ohm’s law, which establishes a local relationship between the surface Laplacian of scalp potentials and the underlying flow of electric current caused by brain activity [see Appendix in Carvalhaes et al., 2014]. Presumably, this local relation should improve spatial resolution, reflecting electrical activity from a more restricted cortical area than what is observed in conventional topography.

In contrast to other high-resolution techniques, such as cortical surface imaging [Nunez et al., 1994, Yao, 1996, Yao et al., 2001], the surface Laplacian has the advantage of not requiring a volume conductor model of the head or a detailed specification of neural sources, but objections to it may arise due to a combination of factors. First, there is a noticeable difficulty in understanding the technique in depth [Nunez and Westdorp, 1994]. While EEG potentials can be physically understood in terms of current flow or in analogy with the gravitational potential, interpreting the surface Laplacian operation seems much less intuitive, as it involves the application of a second-order partial differential operator to a scalar potential distribution. Second, it appears surprising to many that one can obtain a reference-independent quantity from a signal which is seen to have been contaminated by a reference in its very origin [Nunez and Srinivasan, 2006, ch 7]. Third, it is not

possible to guarantee that the theoretical advantages associated to the use of the Laplacian differentiation will be preserved in the practical world of imperfect measurements and finite samples. In fact, reliable estimates of the Laplacian derivation are technically challenging, and computational methods to perform this task are still a subject of research.

The literature dedicated to provide theoretical explanations about the surface Laplacian technique is scarce [see Nunez and Srinivasan, 2006, Tenke and Kayser, 2012]. There are still some gaps that, if filled, can help comprehension and permit a more intuitive view of it. It is the intention of this paper to contribute to this matter. For this purpose, Section 2 provides physical insights on the surface Laplacian technique that are seldom discussed, but that are at the heart of some issues of interest to the EEG researcher. Section 3 focuses on the computational aspects, providing a comprehensive review of selected methods and their advantages and limitations. In Section 3.5 we present the method of estimating the surface Laplacian derivation by means of a matrix transformation, and Section 3.6 focuses on the regularization problem of smoothing splines that significantly affects surface Laplacian estimates.

2. An Overview of the Physics of EEG

Here we discuss the main theoretical ideas behind the use of the surface Laplacian to introduce concepts from the physics of EEG that are often unfamiliar to the EEG researcher, and that have direct relevance to the

surface Laplacian technique. We do not attempt to reproduce the main arguments about the relationship between the surface Laplacian and the dura-surface potential, nor with the Current Source Density (CSD), which are found, for example, in Nunez and Srinivasan [2006] and Tenke and Kayser [2012]. Instead we focus on the more fundamental aspects that are often overlooked.

2.1. Physical interpretation of the surface Laplacian

To better understand the physical meaning of the surface Laplacian, we start with the volume Laplacian, or simply Laplacian, and its relationship to the quasi-static electric field and potential. Here our treatment is standard, and can be found in any standard textbook such as Jackson [1999] or Schey [2004]. Our discussion focus only on aspects that are directly relevant to EEG.

The scalp EEG always refers to the potential with respect to some reference electrode, but from a physics point of view the *electric field* is more fundamental than the potential. To see this, let us recall that the field has a measurable effect: for a particle of charge q , the field $\mathbf{E}(\mathbf{r})$ at the position $\mathbf{r}$ is operationally *defined* by the ratio between the electric force $\mathbf{F}_E$ acting on the charge at $\mathbf{r}$ and the value of q . Here we use the standard notation of representing scalar quantities in italics and vectors in boldface, e.g. q for the electric charge and $\mathbf{E}$ for the electric field, and we also use the symbol “ $\equiv$ ” for an equality that arises from a definition. From this, the field measures

the force per unit of charge, i.e., $\mathbf{E}$ (as a function of the position $\mathbf{r}$) is given by

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{q} \mathbf{F}_E(\mathbf{r}). \quad (1)$$

For simplicity we ignore any temporal dependence of $\mathbf{E}$, but our results are the same, for all practical purposes, in the typical range of frequencies involved in the brain electrical activity [Jackson, 1999, Nunez and Srinivasan, 2006].

Electric fields generated by point charges are described by Coulomb's Law. A consequence of this law is that $\mathbf{E}$ is a conservative field, i.e., the line integral

$$V_{AB} = - \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} \quad (2)$$

between two points $\mathbf{r}_A$ and $\mathbf{r}_B$ is independent of the path along which it is computed. The value V_{AB} has an important physical meaning: it corresponds to the work per unit of charge necessary to be done on a charged particle when it goes through the field $\mathbf{E}$ from position $\mathbf{r}_A$ to $\mathbf{r}_B$ at a constant speed. Thus, V_{AB} is not only measurable, but of practical use. But also of importance is that the path-independence of V_{AB} implies the existence of a function $V(\mathbf{r})$ such that

$$V_{AB} \equiv V(\mathbf{r}_B) - V(\mathbf{r}_A). \quad (3)$$

This function is called the *electric potential* of the field $\mathbf{E}$.

One important relationship between $\mathbf{E}$ and $V(\mathbf{r})$ is given by the gradient operator. The gradient of $V(\mathbf{r})$, denoted $\text{Grad}(V(\mathbf{r}))$, is defined, in

Cartesian coordinates, as

$$\text{Grad}(V(\mathbf{r})) = \frac{\partial V(\mathbf{r})}{\partial x} \hat{\mathbf{i}} + \frac{\partial V(\mathbf{r})}{\partial y} \hat{\mathbf{j}} + \frac{\partial V(\mathbf{r})}{\partial z} \hat{\mathbf{k}}, \quad (4)$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the orthonormal basis vectors. The gradient is the spatial derivative of V at $\mathbf{r}$, and it forms a vector field with the following properties. For a unitary vector $\hat{\mathbf{u}}$, $\text{Grad}(V(\mathbf{r}')) \cdot \hat{\mathbf{u}}$ gives the rate of change of the function V at point $\mathbf{r}'$ in the direction $\hat{\mathbf{u}}$. Thus, $\text{Grad}(V(\mathbf{r}'))$ at $\mathbf{r}'$ is a vector, pointing at the direction where the function V changes the most, whose magnitude is the rate of change. In other words, the direction perpendicular to $\text{Grad}(V(\mathbf{r}'))$ points at the direction of the *isopotential lines*. From this, it is possible to prove that the function $V(\mathbf{r})$ relates to the field by the expression

$$\mathbf{E}(\mathbf{r}) = -\text{Grad}(V(\mathbf{r})), \quad (5)$$

i.e., the field is the negative gradient of the potential.

It is easy to see that $V(\mathbf{r})$ is not uniquely defined, as any function $V'(\mathbf{r})$ given by

$$V'(\mathbf{r}) \equiv V(\mathbf{r}) - V_0, \quad (6)$$

where V_0 is an arbitrary constant, also gives the same differences of potential V_{AB} between $\mathbf{r}_A$ and $\mathbf{r}_B$, and therefore the same gradient (for instance, when we take the limit of $\Delta\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$ very small). As an example, let us consider the electric field from a point particle with charge q situated at position $\mathbf{r}_0$,

which is given by

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3}.$$

It is easy to show that a potential satisfying (2) is given by

$$V(\mathbf{r}) = -\frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_0|}. \quad (7)$$

For another potential $V'(\mathbf{r}) = -\frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_0|} + V_0$, we see at once that its gradient is exactly the same as for $V(\mathbf{r})$, since V_0 would disappear in derivative in (4). However, the potential (7) is often given in textbooks, as it corresponds to a reference in a point infinitely distant from $\mathbf{r}_0$ (i.e. $V \rightarrow 0$ when $|\mathbf{r}| \rightarrow \infty$).

In a volume conductor, such as the brain, the electric field is related to the current density $\mathbf{j}$ by

$$\mathbf{E}(\mathbf{r}) = \rho \mathbf{j}(\mathbf{r}), \quad (8)$$

where $\rho = 1/\sigma$ is the resistivity, given by the inverse of the conductivity σ ¹. Equation (8) is sometimes referred to as the vector form of Ohm's Law. In a medium with resistivity ρ , a map of the potential function may be used to compute $\mathbf{E}(\mathbf{r})$ and, via equation (8), estimate current sources.

Though we are interested in the surface Laplacian, for completeness we

¹Here, for simplicity, we are ignoring the fact that ρ may depend of both on position and time $\rho = \rho(\mathbf{r}, t)$. Furthermore, when the media is anisotropic, $\mathbf{E}(\mathbf{r})$ is not necessarily in the same direction as $\mathbf{j}(\mathbf{r})$, and ρ needs to be a rank-two tensor. However, at the scalp, equation (8) is an excellent approximation to EEG, as the ρ is approximately isotropic in the tangential direction [Rush and Driscoll, 1968, Nicholson and Freeman, 1975, Wolters et al., 2006, Petrov, 2012].

examine the meaning of the Laplacian, which comes from Gauss's Law. In its differential form Gauss's Law is written as

$$\text{Div}(\mathbf{E}) = 4\pi\rho_Q, \quad (9)$$

where ρ_Q is the charge density². Here $\text{Div}(\mathbf{E})$ is the divergence of the electric field $\mathbf{E}$, a linear differential operator acting on $\mathbf{E}$, which in Cartesian coordinates, where the electric field is represented by $\mathbf{E} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}$, takes the form

$$\text{Div}(\mathbf{E}) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}.$$

The divergence of $\mathbf{E}$ can be interpreted as a local measure of the difference between how much field (technically, its flux) gets into an infinitesimal volume surrounding the point where it is computed and how much of this field gets out. If the divergence is zero, the same amount of field that gets in also gets out; if the divergence is negative, more field gets in than out (sink); and if the divergence is positive, more field gets out than comes in (source). For this reason, the divergence of $\mathbf{E}$ is a measure of sources and sinks of the field. Thus, Gauss's law has the interpretation that electric charges are sources of an electric field: if there are no electric charges, the divergence of the field is

²Usually in physics the charge density is represented by ρ , but here we use the subscript Q to distinguish it from the resistivity, defined as $\rho = 1/\sigma$.

zero. Now, from equation (5) we have at once that

$$\text{Div}(\mathbf{E}) = -\text{Div}(\text{Grad}(V)). \quad (10)$$

The divergence of the gradient, $\text{Div}(\text{Grad}(V))$, is defined as the Laplacian³ of V , denoted by $\text{Lap}(V) \equiv \text{Div}(\text{Grad}(V))$. The Laplacian does not have an interpretation that is as straightforward as the ones for the divergence or the gradient. Its meaning is related to the mean value of the potential around the point where it is computed. For example, imagine that $\text{Lap}(V)$ is positive at some point $\mathbf{r}'$ where V has a local minimum (first derivative is zero). Such positive value means that for the nearby points surrounding $\mathbf{r}'$, most of the values of V are greater than $V(\mathbf{r}')$. So, we can interpret $\text{Lap}(V)$ as measuring the relative mean value of the potential around the point⁴. We can also think of the Laplacian of V as simply the divergence of $\mathbf{E}$. Thus, from (10) and (9) it follows that the Laplacian of the potential is proportional to the electric field sources, i.e.,

$$\text{Lap}(V) = -4\pi\rho_Q. \quad (11)$$

The above discussion showed the connection between $\mathbf{E}$ and V . However,

³Notice that, since both Div and Grad are spatial derivative operators, it follows that the Laplacian is a second-order spatial derivative operator, since it is defined as the divergence of the gradient of V .

⁴The reader not comfortable with differential calculus can refer to the finite difference formula (19), which shows that the Laplacian is related to the mean value of the potential around the point.

it did not take into account some characteristics of the EEG. First, the EEG is measured over the scalp, which is a curved surface. Second, such measurements are often not directly of the field, but are differences of potential associated to currents flowing in the circuit formed between two electrodes, the measuring device, and the current flow density in the head⁵. In the head, such currents define a vector field consisting of currents from the brain, dura surface, bone, and scalp. This current density is proportional at each point to the electric field. But outside of the head, the EEG measuring apparatus closes the circuit and measures the small current associated with such system [Metting van Rijn et al., 1990, 1991]⁶. Finally, because such measurements are on the scalp, they happen where there is a significant change in conductivity (interface scalp/air). All of those points have some consequence for the interpretations of the field and Laplacian.

Starting with the Laplacian, the most common assumption is that sources of interest are inside the skull, and that there are no sources in the scalp itself [Nunez and Srinivasan, 2006]⁷. But if we start with this assumption, we have

⁵An attentive reader may object to this statement, arguing that modern EEG apparatuses use operational amplifiers to directly measure the potential. It should be pointed out that, though they are often said to measure potential, the internal circuit of an operational amplifier is simply a high-impedance amplifier, and that they detect the differences of potential by small currents induced in their circuits.

⁶This is the case for standard EEG systems, including very high impedance ones, but not true for capacitive measurements [Spinelli and Haberman, 2010].

⁷This, of course, is an approximation, given that in some EEG setups, the electrodes can chemically generate currents. The discussion of currents generated by the scalp-electrode interface are beyond the scope of this paper [Metting van Rijn et al., 1990, 1991, Huigen et al., 2002, Chi et al., 2010]

that, from (9),

$$\text{Lap}(V) = 0. \quad (12)$$

Using the expression for the Laplacian in Cartesian coordinates, we have⁸

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad (13)$$

Let us now choose the coordinate system such that the scalp is on the plane x, y (for small areas, this is a good approximation). Then we can rewrite (13) as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial E}{\partial z}, \quad (14)$$

where on the right hand side we used the relationship between the field and the potential. Remembering that in a conductor $\mathbf{E} = \rho \mathbf{j}$, it follows that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \rho \frac{\partial j_z}{\partial z}. \quad (15)$$

⁸It may be argued that (12) may not be well-defined on the surface of the scalp because of its discontinuous boundary conditions. First of all, it is important to point out that the Laplacian is related through the right hand side of (12) to a physically measurable quantity, and therefore always defined at a point. Second, the ill-defined character of the Laplacian comes from oversimplified mathematical models of the boundary conditions, and can be eliminated by simply taking the lateral derivative at the interface. For instance, for the equations below, such as (13), if we think about the direction $\hat{\mathbf{z}}$ as perpendicular to the scalp, with increasing values as we leave the scalp, then we can think of the term involving a derivative with respect to z as defined as the left derivative (outward bound). Finally, from a practical point of view, such points where the derivatives are not well defined consist of a set of measure zero, and therefore of little practical importance [see Carvalhaes et al., 2014, for details].

The left hand side of (15) is defined as the *surface Laplacian* of V ,

$$\text{Lap}_S(V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}, \quad (16)$$

and (15) shows that $\text{Lap}_S(V)$ is related to *how abruptly the normal component of the current changes in the direction perpendicular to the surface*. Thus, if $\text{Lap}_S(V)$ is nonzero, conceptually, in the absence of sources, the current change in this direction means a fanning out of the current lines. In the case of the scalp, this means that a nonzero $\text{Lap}_S(V)$ corresponds to *diverging* (with respect to the radial direction) *current lines under the scalp*, which is associated to the presence of a source of currents inside the skull. Thus, the surface Laplacian has a direct relationship to currents under the scalp, and it can be shown that it is related to electrical activities on the dura surface [Nunez and Srinivasan, 2006].

2.2. Properties of the surface Laplacian

With an interpretation of the surface Laplacian in terms of currents and fields, we now turn to some of its properties, starting with its *reference-free* property. This may seem puzzling, as in most cases the surface Laplacian is computed from the scalp potential, which is itself a reference-dependent quantity. To illustrate this issue, let us examine a simple example from geometry. Imagine two events whose locations are given by the points P and Q . From an observer $\mathcal{O}$ (Figure 1), P and Q 's positions are the (reference-dependent) vectors $\mathbf{r}_P$ and $\mathbf{r}_Q$. This dependency on reference can be seen by

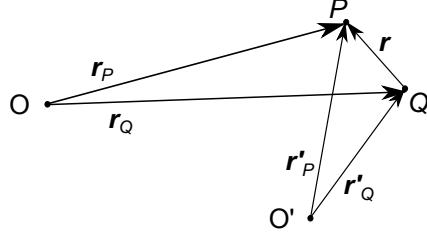


Figure 1: Events P and Q as observed from $\mathcal{O}$ and $\mathcal{O}'$.

the fact that another observer $\mathcal{O}'$ describe such positions by the vectors $\mathbf{r}'_P$ and $\mathbf{r}'_Q$, which are clearly different from $\mathbf{r}_P$ and $\mathbf{r}_Q$. Thus, the geometrical meaning of, say, the length of the position vector $\mathbf{r}_P$, is not only related to the event, but to a combination of such event and an *arbitrary* choice of reference system, making this quantity reference dependent. However, there are many geometrical reference-free quantities that can be constructed from the reference-dependent positions $\mathbf{r}_P$ and $\mathbf{r}_Q$. For example, the vector connecting P and Q is reference-free, since it is given by $\mathbf{r} = \mathbf{r}_P - \mathbf{r}_Q = \mathbf{r}'_P - \mathbf{r}'_Q$. Notice that $\mathbf{r}$ has the feature of depending only on characteristics of the system of interest, P and Q . Another reference-free quantity is the distance d between P and Q , defined by $d^2 = \mathbf{r}^2 = (\mathbf{r}_P - \mathbf{r}_Q) \cdot (\mathbf{r}_P - \mathbf{r}_Q)$. Thus, from reference-dependent geometrical quantities it is possible to obtain reference-free ones.

For the EEG, the reference-free property is not unrelated to the fact that the surface Laplacian (SL) has physical meaning, whereas the scalp potential does not⁹. For instance, in a medium where the electric conductivity σ is

⁹Furthermore, this property is connected to claims that the EEG topography is not

non-zero, such as in the brain, skull, or scalp, the field relates to a (local) current by

$$\mathbf{E} = \rho \mathbf{j}, \quad (17)$$

where $\rho = 1/\sigma$ is the resistivity and $\mathbf{j}$ is the electric current density. The electric potential, on the other hand, is not operationally defined; instead, the *difference* of potential between two points is. For instance, as seen above, in most EEG experiments what is measured is the current between two electrodes, which is proportional to the difference of potential between them. In practice, the potential differences are estimated by the insertion of a circuit in parallel with the electrodes, such that currents in this circuit can be measured [Nunez and Srinivasan, 2006] as a proxy for the difference of potential; the effect of such measurement is minimized by designing a high impedance measuring circuit [Metting van Rijn et al., 1990, 1991]. So, if we define a potential $V(\mathbf{r})$, the value of it tells us nothing about measurable quantities in $\mathbf{r}$; what is measurable is the potential difference between two observation points, $\Delta V = V(\mathbf{r}') - V(\mathbf{r})$. In other words, from a physical point of view it is nonsensical to say that the potential at a point P is $V(\mathbf{r}_P)$; however, it makes sense to say that the electric field at P is $\mathbf{E}(\mathbf{r}_P)$. From the electric field we can obtain the Laplacian, given by $\text{Div}(\mathbf{E})$. Thus, the Laplacian is simply the divergence of $\mathbf{E}$, and as such is nonzero where there are sinks or

affected by changes in the reference electrode, which corresponds to the statement that differences of potential between two points across the topography are invariant with respect to the reference.

sources of electric field.

In practice, directly measuring the scalp field or SL is technically very difficult. However, the usual measurements of differences of potential on the scalp is technically (fairly) easy. It is therefore common to use the observed differences of potential (often defined against a reference electrode) to calculate the SL or the electric field (see Section 3 for computational methods). Being reference free does not mean that the SL of V computed from an EEG is not subject to problems of bad reference-electrode choice (see fNunez and Srinivasan, 2006, for a discussion of the issues involving reference electrodes).

The above discussion holds for all cases where a quasi-static electric field interaction is involved [Hämäläinen et al., 1993]. We included it here because it is seldom present in the EEG review literature, but it bears relevance to many of the concepts often overlooked. We now turn to a description of the computational methods.

3. Computational Methods

EEG experiments are ordinarily designed to record the electric potential, and any information about its spatial derivatives, including the SL, needs to be estimated from the recordings. There are several methods available for SL estimates, but some may be preferable to others for particular purposes. The first Laplacian estimates in the literature were performed by

Hjorth, 1975¹⁰, who presented an approximation to equation (16) based on *finite difference*. This method is conceptually simple and easy to implement, and for this reason still very popular. It also does not suffer from some numerical problems that affect its main alternatives, but has the downside of relying on assumptions that are quite questionable empirically. In a further development, *mesh-free* methods [Perrin et al., 1987b,a, 1989] were introduced, eliminating problems encountered in finite difference, but other issues emerged, some of which still need attention. Our goal here is to review both types of approaches and emphasize issues related to each.

Two other topics are discussed in detail. The first is the *SL matrix*, which significantly reduces the computational cost of estimating the SL. The reductions in computational cost are large and should not be overlooked, mainly when using mesh-free methods. The second topic is related to the procedure of *regularization* to increase the quality of estimates. Besides reviewing this subject and presenting an example, we also suggest a solution to an important problem arising in the regularization of multiple frames of an EEG signal.

3.1. Finite difference methods

Generally speaking, finite difference is a *discretization procedure* in which a continuous domain is transformed into a mesh of discrete points, where

¹⁰We also acknowledge the work of Freeman and Nicholson [1975], who employed a discretization scheme to approximate the Laplacian of extra-cellular potentials in CSD analysis.

differential operations such as the SL become manageable. Figure 2 depicts the construction used here. In it, each electrode occupies an individual node of a regular grid specified by the discrete variables i and j , in replacement to the continuous variables x and y used before. Two major assumptions are made. First, the scalp surface is locally approximately flat, and second the measuring electrodes are equidistant, forming a square grid of size h . The first assumption leads to the following differential form in Cartesian coordinates (see also Section 2.1)

$$\text{Lap}_s(V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}. \quad (18)$$

With the second assumption we approximate $\text{Lap}_s(V)$ at a central node (i, j) by [Abramowitz and Stegun, 1964, 25.3.30]

$$\text{Lap}_s(V)|_{(i,j)} \approx \frac{V_{(i-1,j)} + V_{(i+1,j)} + V_{(i,j-1)} + V_{(i,j+1)} - 4V_{(i,j)}}{h^2}. \quad (19)$$

The mathematics behind this approximation is nontrivial and for completeness we include it in Appendix A.

As expected from Section 2.2, this approximation is reference-free. To see this, replace V with $V' - V_{\text{ref}}$ at each node (including the central node) and note that the common reference V_{ref} is canceled. The right hand side of (19) can be interpreted as a change in reference to the average over the

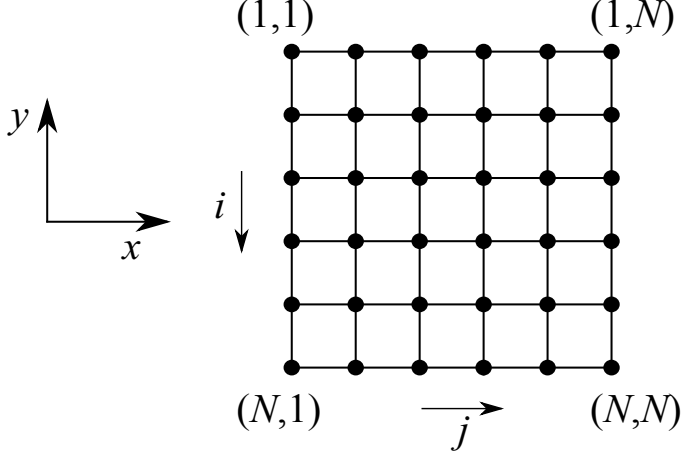


Figure 2: Mapping from the continuous variables x and y into the grid variables i and j .

four nearest neighbors of node (i, j) .¹¹ If the reference electrode is located on the scalp we can reconstruct the Laplacian at this location, where $V = 0$, by computing

$$\text{Lap}_s(V_{\text{ref}}) \approx \frac{V_{(i-1,j)} + V_{(i+1,j)} + V_{(i,j-1)} + V_{(i,j+1)}}{h^2}. \quad (20)$$

The multiplicative factor $1/h^2$ in (19) and (20) ensures the correct physical unit of $\text{Lap}_s(V)$, which is Volt per centimeter square (V/cm^2), but in many situations this is ignored, as originally done in Hjorth [1975].

Approximation (19) is usually referred to as *Hjorth's approximation*. A limitation of Hjorth's approximation is that it applies only to electrodes located at a central node, thus not accounting for estimates along the border

¹¹Notice that this reference change happens locally, i.e. it is different for each electrode position, which means that the Laplacian is not simply another reference scheme.

of the electrode grid. For border electrodes, Hjorth suggested a less accurate approximation using just three electrodes aligned along the border. In an effort to account for estimations at all electrode sites, we developed the scheme of Figure 3. For instance, estimates along the left border, where $j = 1$, are given by

$$\text{Lap}_s(V_{(i,1)}) \approx \frac{V_{(i-1,1)} + V_{(i+1,1)} - 2V_{(i,2)} + V_{(i,3)} - V_{(i,1)}}{h^2}, \quad (21)$$

and at the upper-left corner

$$\text{Lap}_s(V_{(1,1)}) \approx \frac{-2V_{(1,2)} + V_{(1,3)} - 2V_{(2,1)} + V_{(3,1)} + 2V_{(1,1)}}{h^2}. \quad (22)$$

Our calculations are shown in Appendix A. As it occurs with approximation (19), these expressions average the potential with weights that sum to zero, thus canceling the reference potential.

The procedure described in Appendix A can be used to obtain approximations combining more than five electrodes. However, increasing complexity does not result necessarily into more accurate estimates. Approximations for unevenly-spaced electrodes can also be derived, but we will not address this here [cf. Tenke et al., 1998].

Finite difference provides a measure of accuracy in terms of the discretization parameter h . As explained in Appendix A, approximation (19) is said to be accurate to the second-order of h , shorthand $\mathcal{O}(h^2)$, while for peripheral electrodes the five-point approximations (21) and (22) are accurate

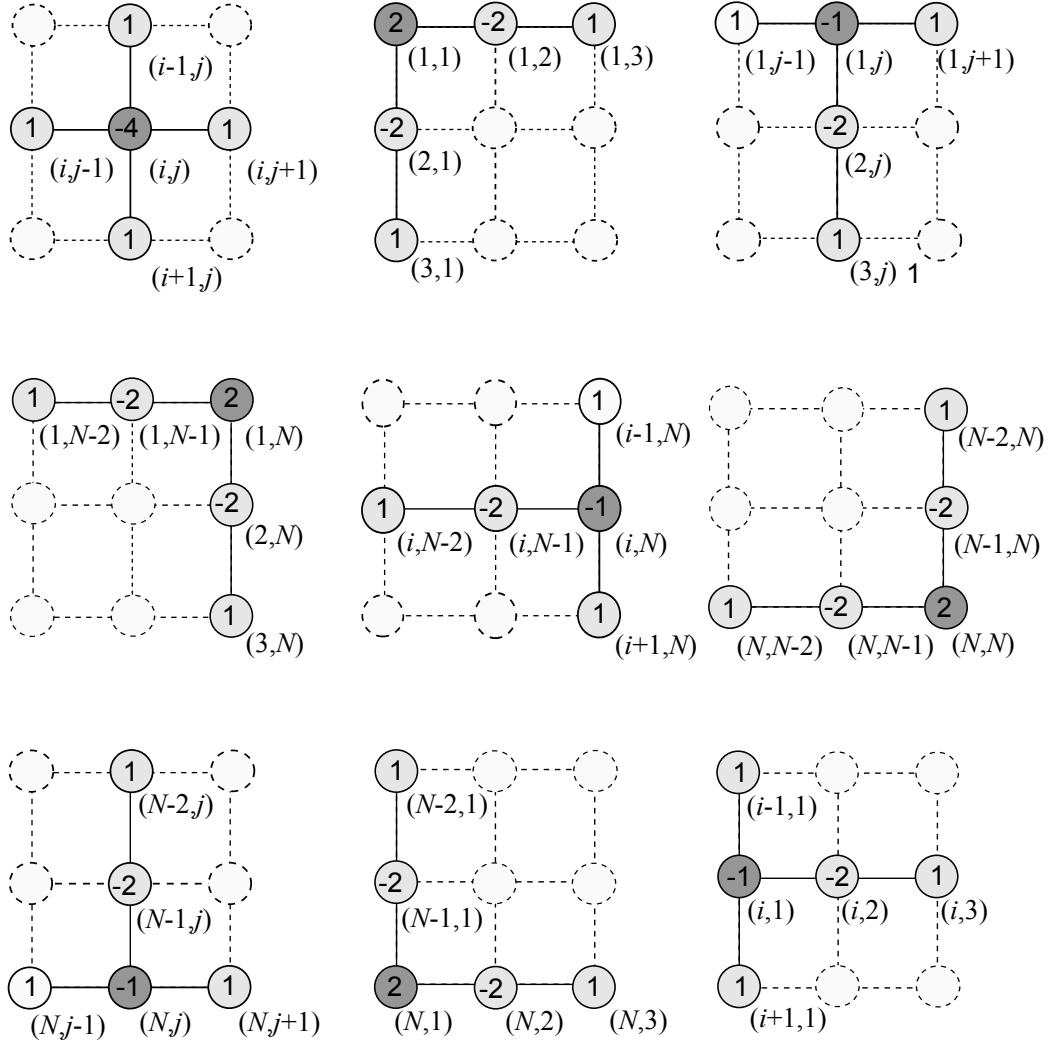


Figure 3: Geometric arrangements and weights to approximate $\text{Lap}_s(V)$ at different locations on a grid.

to $\mathcal{O}(h)$. The higher the order, the more accurate is the approximation [Abramowitz and Stegun, 1964]. Therefore, in principle, making h small by using high-density electrode arrays should reduce discretization error and improve estimates. Moreover, a small h is also useful to reduce error due to the unrealistic geometry of the plane scalp model, which is more salient over sparse electrode sites. In the absence of an appropriate mechanism for regulation, high-density arrays can also help reduce influence of non-local activities from unrelated sources. But exceptions are expected to occur. In fact, all this reasoning was based on the assumption of a point electrode, which ignores boundary effects produced by an actual electrode of finite size. Perhaps for this reason, McFarland et al. [1997] reported better estimates using next-nearest-neighbor approximations in comparison to nearest-neighbor approximations. Similar conclusion was drawn by Müller-Gerking et al. [1999] in the context of EEG classification. In fact, next-nearest neighbor approximations have been adopted in CSD analysis since Freeman and Nicholson [1975] as a way to reduce high-frequency noise in the vicinity of an electrode. The trade-off between noise attenuation and loss of spatial resolution in low-density arrays was discussed by Tenke et al. [1993]. In fact, lower-density estimates has proven to be quite useful more than once, as described by Kayser and Tenke [2006a] in the context of ERP analysis. Finally, we point out that evidence for electrode bridges in high-density EEG recordings was identified as a problem caused typically by electrolyte spreading between nearby electrodes [Tenke and Kayser, 2001, Greischar et al., 2004, Alschuler

et al., 2014].

3.2. Smoothing thin-plate spline Laplacians

As an alternative to finite difference, *mesh-free* (or *grid-free*) methods allow for a more flexible configuration of electrodes and are not restricted to the planar scalp model. In this case, the Laplacian differentiation is performed analytically on a continuous function built from the data. Two ways of obtaining this function are through interpolation or *data smoothing*. In the context of interpolation, the problem to be solved can be formulated as follows: given a set of measurements, say a potential $V_1, \dots, V_N$ recorded at locations $\mathbf{r}_1, \dots, \mathbf{r}_N$ (the electrode locations), find a real function $f(\mathbf{r})$ such that $f(\mathbf{r}_i) = V_i$ for $i = 1, \dots, N$. For data smoothing, this constraint is replaced with the condition that $f(\mathbf{r}_i)$ should fit the data closely, but not exactly, to reduce variance due to noise. The method of *thin-plate splines* discussed here is a prime tool for both cases.

Formally, the spline function is the *unique* solution to the problem of finding a function f that minimizes [Duchon, 1977, Meinguet, 1979, Wahba, 1990, Hastie et al., 2009, p.140]

$$\text{RSS}(f, \lambda) = \frac{1}{N} \sum_i (V_i - f(\mathbf{r}_i))^2 + \lambda J_m[f(\mathbf{r})], \quad (23)$$

where it was assumed that $\mathbf{r}_1 \neq \mathbf{r}_2 \neq \dots \neq \mathbf{r}_N$ (which is always true for actual electrodes) and $J_m[f]$ is a measure of roughness f in terms of its m -th-order partial derivatives [cf. Wahba, 1990, for details]. The parameter

λ is called the *regularization parameter*. This parameter trades off between goodness of fit ($\lambda = 0$) and smoothness ($\lambda > 0$). In other words, if $\lambda = 0$ then the interpolation condition $f(\mathbf{r}_i) = V_i$ is satisfied, whereas for $\lambda > 0$ the spline function is smooth and violates the interpolation condition. In three-dimensional Euclidean space the solution to this problem has the form [Wahba, 1990, Wood, 2003]

$$f_\lambda(\mathbf{r}) = \sum_{i=1}^N c_i |\mathbf{r} - \mathbf{r}_i|^{2m-3} + \sum_{\ell=1}^M d_\ell \nu_\ell(\mathbf{r}), \quad (m \geq 2). \quad (24)$$

The parameters m and M are integer numbers satisfying $M = \binom{m+2}{3}$ and $M < N$. In addition, m must be greater than 2 for the SL differentiation to be well-defined. The functions $\nu_1, \dots, \nu_M$ are linearly independent polynomials in three variables of degree less than m [cf. Carvalhaes and Suppes, 2011, for computational details].

Using matrix notation we can express the unknowns c_i 's and d_i 's as the solution to the linear system [Duchon, 1977, Meinguet, 1979, Wahba, 1990, Green and Silverman, 1994, Eubank, 1999]

$$\begin{pmatrix} \mathbf{K} + N\lambda\mathbf{I} & \mathbf{T} \\ \mathbf{T}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix}, \quad (25)$$

where $\mathbf{c} = (c_1, \dots, c_N)^T$, $\mathbf{d} = (d_1, \dots, d_M)^T$, $(\mathbf{K})_{ij} = |\mathbf{r}_j - \mathbf{r}_i|^{2m-3}$, $(\mathbf{T})_{ij} = \nu_j(\mathbf{r}_i)$, and $\mathbf{v} = (V_1, \dots, V_N)^T$ is the given potential distribution. The superscript T indicates transpose operation and $\mathbf{I}$ is the $N \times N$ identity matrix.

This system has the formal solution [Wahba, 1990]

$$\mathbf{c} = \mathbf{Q}_2 [\mathbf{Q}_2^T (\mathbf{K} + N\lambda\mathbf{I}) \mathbf{Q}_2]^{-1} \mathbf{Q}_2^T \mathbf{v}, \quad (26a)$$

$$\mathbf{R}\mathbf{d} = \mathbf{Q}_1^T (\mathbf{v} - \mathbf{K}\mathbf{c} - N\lambda\mathbf{c}), \quad (26b)$$

where $\mathbf{Q}_1 \in \mathbb{R}^{N \times N}$, $\mathbf{Q}_2 \in \mathbb{R}^{N \times (N-M)}$, and $\mathbf{R} \in \mathbb{R}^{N \times M}$ derive from the *QR-factorization* of $\mathbf{T}$ ¹²:

$$\mathbf{T} = [\mathbf{Q}_1 \mid \mathbf{Q}_2] \begin{pmatrix} \mathbf{R} \\ \mathbf{0} \end{pmatrix}. \quad (27)$$

With the above definitions we can express the smoothed potential at the electrode locations as

$$\mathbf{v}_\lambda = \mathbf{K}\mathbf{c} + \mathbf{T}\mathbf{d}. \quad (28)$$

By introducing the matrices $(\tilde{\mathbf{K}})_{ij} = \text{Lap}(|\mathbf{r}_j - \mathbf{r}_i|^{2m-3})$ and $(\tilde{\mathbf{T}})_{ij} = \text{Lap}(\nu_j(\mathbf{r}_i))$, we obtain an analogous expression for the Laplacian estimate, which is

$$\text{Lap}_S(\mathbf{v}) = \tilde{\mathbf{K}}\mathbf{c} + \tilde{\mathbf{T}}\mathbf{d}. \quad (29)$$

This approach has a major deficiency: it does not work if the data points $\mathbf{r}_1, \dots, \mathbf{r}_N$ are located on a spherical or an ellipsoidal surface. To understand this, consider the case $m = 3$, for which the matrix $\mathbf{T}$ can be expressed as

¹²The Matlab command for the QR-factorization of $\mathbf{T}$ is $[\mathbf{Q}, \mathbf{R}] = \text{qr}(\mathbf{T})$.

[Carvalhaes and Suppes, 2011]

$$\mathbf{T} = \begin{pmatrix} 1 & x_1 & y_1 & z_1 & x_1^2 & x_1 y_1 & x_1 z_1 & y_1^2 & y_1 z_1 & z_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & y_N & z_N & x_N^2 & x_N y_N & x_N z_N & y_N^2 & y_N z_N & z_N^2 \end{pmatrix}. \quad (30)$$

Since on spherical and ellipsoidal surfaces the coordinates x_i , y_i , and z_i are subject to¹³

$$\frac{x_i^2}{a^2} + \frac{y_i^2}{b^2} + \frac{z_i^2}{c^2} = 1, \quad a, b, c > 0, \quad (31)$$

the 1st, 5th, 8th, and 10th columns of $\mathbf{T}$ are *linearly dependent*¹⁴. Hence, we can always express one of these columns in terms of the others. This is equivalent to say that $\mathbf{T}$ has 10 columns but only 9 are linearly independent. This makes the linear system (25) *singular*, so that the unknowns c_i 's and d_i 's cannot be uniquely determined.

The singularity of (25) on spheres and ellipsoids affects only the transformation parametrized by $\mathbf{d}$, leaving intact the transformation specified by $\mathbf{c}$. It is therefore natural to try a minimum norm solution to this problem by determining $\mathbf{d}$ using the *pseudo-inverse* of $\mathbf{R}$, i.e., $\mathbf{d} = \mathbf{R}^+ \mathbf{Q}'_1 (\mathbf{v} - \mathbf{K}\mathbf{c} - N\lambda\mathbf{c})$. This approach was proposed by Carvalhaes and Suppes [2011] (see also Carvalhaes, 2013) and was evaluated in a problem involving the localization of cortical activity on spherical and ellipsoidal scalp models. Simulations using

¹³Note that $a = b = c$ on the sphere.

¹⁴That is, summing these columns with weights -1 , $1/a^2$, $1/b^2$ and $1/c^2$ yields a zero vector.

over 30,000 configurations of radial dipoles resulted in a success rate above 94.5% for the correct localization of cortical source at the closest electrode. This rate improved to 99.5% for the task of locating the electrical source at one of the two closest electrode. Prediction error occurred more often for sources generating very small or very large peaks in the amplitude of the potential, but it decreased substantially with increasing the number of electrodes in the simulation. In the same study, but now using empirical data, the SL of (24) outperformed finite difference and the method of spherical splines discussed below.

Similar to finite difference, splines and other methods discussed below are usually affected by the density of the electrode array. In general, high-density arrays result in more accurate estimates. However, some preparation issues such as electrolyte bridges are mainly a characteristic of high-density estimates and should be a concern, as it is for finite difference [Tenke and Kayser, 2001]. Another concern is that by increasing the number of electrodes the system (25) becomes increasingly sensitive to error (noise) in the input vector $\mathbf{v}$. According to results in Sibson and Stone [1991] it is expected that montages with up to 256 electrodes result in reliable estimates for double-precision calculations, but there is one more issue requiring special attention.

It follows from (26a) that in order to obtain the coefficient vector $\mathbf{c}$ the matrix $[\mathbf{Q}_2^T (\mathbf{K} + N\lambda\mathbf{I}) \mathbf{Q}_2]$ needs to be inverted. However, it turns out that this inverse operation is accurate only if the condition number of $(\mathbf{K} + N\lambda\mathbf{I})$

is relatively small.¹⁵. In principle, a condition number up to 10^{12} should be acceptable for splines [Sibson and Stone, 1991], but this upper bound can easily be exceeded as λ decreases toward zero depending on the electrode configuration. In fact, even for a small number of electrodes the condition number of $(\mathbf{K} + N\lambda\mathbf{I})$ can be so high that is impossible to invert $[\mathbf{Q}_2^T (\mathbf{K} + N\lambda\mathbf{I}) \mathbf{Q}_2]$ without regularization. This problem holds also for the algorithm of spherical splines discussed in the next section, and Bortel and Sovka [2007] discussed the same problem in the context of realistic Laplacian techniques. In addition to compensating for conditioning issues, the regularization parameter λ determines also the ability of smoothing splines to account for spatial noise in the data. This problem will be addressed in Section 3.6.

3.3. Smoothing spherical splines

For the case of data points on spheres, Wahba [1981b] developed a pseudo-spline method that circumvent the singularity of (25) by replacing the Euclidean distance with the geodesic distance. This method, called *spherical splines*, was used by Perrin et al. [1989] to develop one of the most popular SL methods in the literature. In this case, the smoothing/interpolating

¹⁵The condition number of a matrix $\mathbf{A}$ is a real number that estimates the loss of precision in inverting this matrix numerically. This number, denoted by $\text{cond}(\mathbf{A})$, is usually computed as the ratio of the largest to smallest eigenvalue of $\mathbf{A}$, implying that $\text{cond}(\mathbf{A}) \geq 1$. The larger the value of $\text{cond}(\mathbf{A})$, the more inaccurate the inversion operation.

function is defined as

$$f_{sph}(\mathbf{r}) = \sum_{i=1}^N c_i g_m(\mathbf{r}, \mathbf{r}_i) + d, \quad (32a)$$

where the parameters $c_1, \dots, c_N$ and d are determined by (26) and

$$g_m(\mathbf{r}, \mathbf{r}_i) = \frac{1}{4\pi} \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell^m (\ell+1)^m} P_{\ell}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_i). \quad (32b)$$

The vectors $\mathbf{r}$ and $\mathbf{r}_i$'s have length equal to the head radius and carets denote unit vectors, i.e., $\hat{\mathbf{r}} = \mathbf{r}/r$ and $\hat{\mathbf{r}}_i = \mathbf{r}_i/r$, so that the scalar product $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_i$ gives the cosine of the angle between $\mathbf{r}$ and $\mathbf{r}_i$. The parameter m is an integer larger than 1, and the P_{ℓ} 's are Legendre polynomials of degree ℓ . These polynomials are common in mathematical physics and their properties are well known [e.g., Abramowitz and Stegun, 1964].

The summation over Legendre polynomials generates progressively higher spatial frequencies which are weighted by the factor $(2\ell+1)/\ell^m(\ell+1)^m$. Since this factor decreases with ℓ , it acts as a Butterworth filter that downweights high-frequency spatial components.

To apply the SL to (32) we use spherical coordinates. Our convention is that of Figure 4: the polar angle $\theta \in [0, \pi]$ is measured down from the vertex, and the azimuthal angle $\varphi \in [0, 2\pi]$ increases counterclockwise from the x

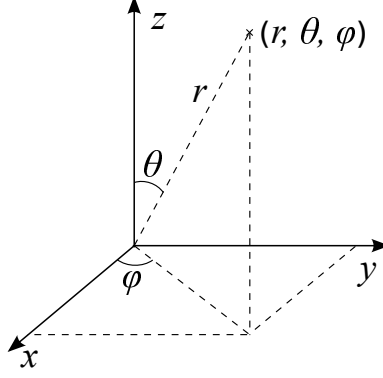


Figure 4: Spherical coordinate system.

axis, which is directed towards the nasion. With this convention,

$$\text{Lap}_s(f) = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial f}{\partial \theta} \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}. \quad (33)$$

It can be proved that the Legendre polynomials satisfy [Jackson, 1999, p. 110]

$$\text{Lap}_s(P_\ell(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_i)) = -\frac{\ell(\ell+1)}{r^2} P_\ell(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_i), \quad (34)$$

thus leading to

$$\text{Lap}_s(f(\mathbf{r})) = -\frac{1}{r^2} \sum_{i=1}^N c_i g_{m-1}(\mathbf{r}, \mathbf{r}_i), \quad m > 1, \quad (35)$$

which is known as the *spherical SL* of $f(\mathbf{r})$.

In comparison to the previous approach, the method of spherical splines has the advantage of providing a simple expression for the SL of $f(\mathbf{r})$ restricted to spherical geometry. Additionally, the geodesic distance ensures

that the system (25) is non-singular, so that the coefficient d can be determined without using the pseudo-inverse of $\mathbf{R}$. However, some caution is required. In practice the term $g_{m-1}(\mathbf{r}, \mathbf{r}_i)$ in (35) needs to be truncated for evaluation, and truncation error should not be overlooked. The left-out terms in the truncation are Legendre polynomials of higher-degree (large ℓ), accounting for high-frequency spatial features. Too much filtering of such features can dilute the potential for improvement in spatial resolution. In view of that, one should keep as many Legendre polynomials as possible. Another point worth considering is the factor $(2\ell+1)/\ell^m(\ell+1)^m$ that multiplies the Legendre polynomials in equation (32b). This factor falls off as $1/\ell^{2m-1}$. As ℓ increases, it approaches zero rapidly, resulting in a quick cancellation of higher-degree polynomials. The value assigned to m must take this effect into account; the larger the value of m , the more dramatic is the reduction in high-frequency features. Typically, a value of m between 2 and 6 provides satisfactory results for simulation and data analysis [Babiloni et al., 1995, de Barros et al., 2006, Carvalhaes and Suppes, 2011].

3.4. Other approaches

There are many other possible ways to estimate the SL of EEG signals. In this section we briefly review the methods proposed by Yao [2002] and Nunez and collaborators [Law et al., 1993a, Nunez and Srinivasan, 2006]. Similar to thin-plate splines, Yao's approach uses interpolation based on radial functions to estimate the SL of the potential. The interpolating function

has the general form

$$f_{\text{RBF}}(\mathbf{r}) = \sum_{i=1}^N c_i e^{-\frac{m}{\pi^2} S(\mathbf{r}, \mathbf{r}_i)} + b, \quad (36)$$

where $S(\mathbf{r}, \mathbf{r}_j)$ measures the arc of circle connecting $\mathbf{r}$ and $\mathbf{r}_i$ and the parameters c_i and b are subject to

$$f_{\text{RBF}}(\mathbf{r}_i) = V_i, \quad i = 1, \dots, N, \quad (37)$$

where $\mathbf{r}_i$ is the location of the i th electrode and V_i is the value of the potential at that location. Because the system (37) contains $N + 1$ unknowns¹⁶ but only N equations, it is solved using pseudo-inversion [see the Appendix in Zhai and Yao, 2004].

In contrast to thin-plate splines, the Gaussian $e^{-\frac{m}{\pi^2} S(\mathbf{r}, \mathbf{r}_i)}$ approaches zero asymptotically as the distance increases from $\mathbf{r}_i$. The parameter m , commonly referred to as the *spread parameter*, controls the rate of decay of $e^{-\frac{m}{\pi^2} S(\mathbf{r}, \mathbf{r}_i)}$. A large m produces a sharp decay, resulting in a short range of influence for each node $\mathbf{r}_i$. Yao recommended to set m according to the number of electrodes in the montage; for instance, $m = 20$ for arrays with 32 sensors, $m = 40$ for 64 sensors, and $m = 50$ for 128 sensors. But he also remarked that a proper choice of m should take into account the source location. In principle, small values of m would be more suitable to fit deep

¹⁶Namely, we have to determine the $N + 1$ parameters $c_1, \dots, c_N$ and d .

brain sources, whereas shallow sources would be better described by small m -values. In other words, interpolation with Gaussian functions is not automatic. The parameter m needs to be tuned properly for good performance, which can be difficult to achieve for non-equidistant electrodes. Nevertheless, Yao showed consistent results favoring the Gaussian method against spherical splines for simulated and empirical data. More recently Bortel and Sovka [2013] questioned the regularization technique used by Yao [2002] and the impossibility of obtaining good estimates without a prior knowledge about the source depth.

The method developed by Nunez and collaborators is called the *New Orleans Spline-Laplacian* [Law et al., 1993a, Nunez and Srinivasan, 2006]. It uses an interpolating function that resembles two-dimensional splines but with knots in $\mathbb{R}^3$ instead of $\mathbb{R}^2$, i.e.,

$$\begin{aligned} f_{\text{NOSL}}(\mathbf{r}) = & \sum_{i=1}^N c_i |\mathbf{r} - \mathbf{r}_i|^4 \log(|\mathbf{r} - \mathbf{r}_i|^2 + \omega^2) + d_1 + d_2x + d_3y \\ & + d_4z + d_5x^2 + d_6xy + d_7xz + d_8y^2 + d_9yz + d_{10}z^2, \end{aligned} \quad (38)$$

where the parameters c_i 's and d_i 's are determined in the fashion of equation (25). This method was implemented for spherical and ellipsoidal scalp models and its performance was studied using simulations and real data [Law et al., 1993a]. A remark about the function $f_{\text{NOSL}}(\mathbf{r})$ is that it has no optimality property, except that for ω and z equal to zero it corresponds to the unique minimizer of (23) in $\mathbb{R}^2$ [Duchon, 1977, Meinguet, 1979]. But $f_{\text{NOSL}}(\mathbf{r})$

does not minimize (23) in $\mathbb{R}^3$, as the minimizer of (23) in $\mathbb{R}^3$ is unique and correspond to the spline function (24). In other words, $f_{\text{NOSL}}(\mathbf{r})$ is not actually a spline function.¹⁷ Despite that, Nunez and colleagues reported many studies showing a good performance of their method [Law et al., 1993b, Nunez and Westdorp, 1994, Nunez et al., 2001, Nunez and Srinivasan, 2006] and emphasized its agreement with cortical imaging algorithms in terms of correlation [Nunez et al., 1993, 1994].

3.5. The surface Laplacian matrix

The computational cost of estimating the SL is usually not prohibitive, but in many circumstances it enough to justify the use of an additional technique to improve performance. The *SL matrix* $\mathbf{L}$ is an excellent tool for this, and because of the linearity of the SL differentiation, it can be created for any of the SL methods above. For interpretation purposes the SL matrix can be viewed as a discrete representation of the differential operator Lap_S at the electrode sites. We will illustrate below the method for finite difference and splines. Once the matrix $\mathbf{L}$ is obtained the SL estimate is carried out through the linear transformation

$$\text{Lap}_S(\mathbf{v}) = \mathbf{L}\mathbf{v}. \quad (39)$$

¹⁷Due to the constraint (31), the NOSL algorithm is also affected by the singularity of (25) on spherical and ellipsoidal scalp models (see discussion in Section 3.2). Apparently, in a effort to remedy this problem the Matlab code of Nunez and Srinivasan [2006, p. 585] adds “noise” (error) to zero-valued electrode coordinates, effectively making (25) nonsingular, but not necessarily leading to a well-conditioned solution.

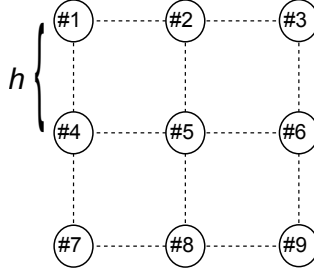


Figure 5: A montage with 9 electrodes in a square grid of spacing h . The numbers 1 to 9 identify the electrodes and should not be confused with the weights for a finite difference approximation.

An important aspect of $\mathbf{L}$ is that its construction does not involve brain data, as otherwise it could not represent the mathematical operator Lap_S . This way we need to create $\mathbf{L}$ only once for each method and electrode configuration. But notice that $\mathbf{L}$ depends on the parameters m and λ for splines.

We will first illustrate the construction of $\mathbf{L}$ for finite difference approximations. In this case, $\mathbf{L}$ is constructed row by row using the scheme of Figure 3. Electrodes that do not contribute to an approximation are assigned a zero weight in that computation. This way, each row of $\mathbf{L}$ has as many elements as the number of channels in the montage, but only five elements are nonzero in each row. This implies that $\mathbf{L}$ is a sparse matrix, with its sparsity increasing with the number of channels. Because the weights of each approximation sum to zero, the columns of $\mathbf{L}$ also sum to zero, which reflects the fact that the Laplacian transformation is reference-free.

As a simple illustration, consider the electrode arrangement of Figure 5, with 9 electrodes evenly distributed in a square grid of spacing h .

The Laplacian matrix of this arrangement has dimension 9×9 and is given by

$$\mathbf{L} = \frac{1}{h^2} \begin{pmatrix} 2 & -2 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -2 & 0 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & 0 & -2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 & -2 & 2 \end{pmatrix}. \quad (40)$$

A quick inspection on the diagonal elements of $\mathbf{L}$ reveals that none of them is equal to zero, but they sum to zero. This means that $\mathbf{L}$ *does not possess inverse*. The inability to invert $\mathbf{L}$ is not exclusive of finite differences, but rather a general property that reflects the ambiguity on the choice of reference. Stated in other words, *the Laplacian transformation (39) cannot be uniquely undone due to the fact that the potential is not uniquely defined*.

Mesh-free methods generate $\mathbf{L}$ matrices that are non-sparse, i.e., most of their entries are nonzero. In the context of smoothing splines, Carvalhaes and Suppes [2011] showed that

$$\mathbf{L}_\lambda = \tilde{\mathbf{K}} \mathbf{C}_\lambda + \tilde{\mathbf{T}} \mathbf{D}_\lambda, \quad (41)$$

where the subscript λ emphasizes dependency on the regularization parameter, and

$$(\tilde{\mathbf{K}})_{ij} = \text{Lap}_s(|\mathbf{r}_i - \mathbf{r}_j|^{2m-3}), \quad (42a)$$

$$(\tilde{\mathbf{T}})_{ij} = \text{Lap}_s(\nu_i(\mathbf{r}_j)), \quad (42b)$$

$$\mathbf{C}_\lambda = \mathbf{Q}_2 [\mathbf{Q}_2^T (\mathbf{K} + N\lambda\mathbf{I}) \mathbf{Q}_2]^{-1} \mathbf{Q}_2^T, \quad (42c)$$

$$\mathbf{D}_\lambda = \mathbf{R}^+ \mathbf{Q}_1^T (\mathbf{1} - \mathbf{K} \mathbf{C}_\lambda - N\lambda \mathbf{C}_\lambda). \quad (42d)$$

The matrix $\mathbf{L}_\lambda$ also has no inverse, thus not allowing unambiguous inverse operation. This is due to the fact that $\mathbf{C}_\lambda$ and $\tilde{\mathbf{T}}$ are rank-deficient. In other words, since $\mathbf{Q}_2$ has only $N - 1$ columns, only $N - 1$ degrees of freedom are available from $\mathbf{C}_\lambda$. In turn, because the Laplacian operation $\text{Lap}_s(\nu_i(\mathbf{r}_j))$ vanishes at any polynomial ν_i of degree less than 2, the first four columns of $\tilde{\mathbf{T}}$ are null.

For spherical splines, $(\tilde{\mathbf{K}})_{ij} = -g_m(\mathbf{r}_j, \mathbf{r}_i)/r^2$, $\mathbf{T}$ is a N -vector of ones, and $\tilde{\mathbf{T}}$ is null, so that

$$\mathbf{L}_\lambda = \tilde{\mathbf{K}} \mathbf{C}_\lambda, \quad (43)$$

which has rank $N - 1$ and does not possess inverse. Since $\mathbf{T}$ is a vector of ones, its QR-factorization gives $\mathbf{R} = -\sqrt{N}$, i.e., $\mathbf{R}$ is a scalar instead of a matrix, and $\mathbf{Q}_1 = -\mathbf{T}/\sqrt{N}$. Consequently, $\mathbf{D}_\lambda = \mathbf{T}^T (\mathbf{1} - \mathbf{K} \mathbf{C}_\lambda - N\lambda \mathbf{C}_\lambda) / N$, which is also a N -vector instead of a matrix.

As anticipated, the construction of $\mathbf{L}$ and $\mathbf{L}_\lambda$ is carried out without using

brain data. Because the matrix $\mathbf{C}_\lambda$ depends on the inverse of the matrix $[\mathbf{Q}_2^T (\mathbf{K} + N\lambda\mathbf{I}) \mathbf{Q}_2]$ it is subject to the numerical issues discussed in Section 3.2, so that it is recommended to set λ nonzero and keep m as small as possible for accurate results. The fact that the matrix $\mathbf{L}$ for finite difference is sparse implies that the computational cost of finite difference is generally much smaller than the cost of splines. A quick calculation shows that for a signal containing τ time frames the cost of computing (39) with finite difference is $5 N \tau$ floating-point operations (flops)¹⁸, while for splines this amounts to $N^2 \tau$ flops. If we decide not to use the matrix $\mathbf{L}$, then the system (25) needs to be solved once for each frame, at the total cost of $(2/3) N^3 \tau$ flops.¹⁹

3.6. The regularization of smoothing splines

The use of smoothing splines to estimate the SL requires to fix the regularization parameter λ , which, as said above, has the beneficial effect of eliminating spatial noise. A key factor to guide towards a good choice of λ is the mean square error function

$$R(\lambda) = \frac{1}{N} \sum_{i=1}^N (V_i - f_\lambda(\mathbf{r}_i))^2. \quad (44)$$

Intuitively, this function measures infidelity of $f_\lambda(\mathbf{r})$ to the input data $(V_1, \dots, V_N)$ due to smoothing. The goal of regularization is to weight this function appropriately to achieve a good balance between fidelity and variance reduc-

¹⁸This formula is valid only if $\mathbf{L}$ is stored in the computer as a sparse matrix.

¹⁹This is valid for an efficient resolution using LU decomposition [Press, 1992].

tion. The popular method of *generalized cross-validation* (GCV) proposed by Craven and Wahba [1979] addresses this problem in the following way. The GCV estimate of λ is the value for which the GCV function

$$\text{GCV}(\lambda) = \frac{1}{N} \frac{\sum_{i=1}^N (V_i - f_\lambda(\mathbf{r}_i))^2}{(1 - \text{trace}(\mathbf{S}_\lambda)/N)^2}, \quad (45)$$

reaches its minimum value. That is, the GCV criterion is to correct the squared residuals $(V_i - f_\lambda(\mathbf{r}_i))^2$ about the estimate by dividing them by the factor $(1 - \text{trace}(\mathbf{S}_\lambda)/N)^2$. The matrix $\mathbf{S}_\lambda$ is a square matrix satisfying

$$\begin{pmatrix} f_\lambda(\mathbf{r}_1) \\ \vdots \\ f_\lambda(\mathbf{r}_N) \end{pmatrix} = \mathbf{S}_\lambda \begin{pmatrix} V_1 \\ \vdots \\ V_N \end{pmatrix}. \quad (46)$$

The matrix $\mathbf{S}_\lambda$ is called the *smoother matrix*. The value $\lambda = 0$ (no smoothing) is excluded from GCV predictions, as it leads to $\text{trace}(\mathbf{S}_\lambda) = N$. It follows from our previous development that

$$\mathbf{S}_\lambda = \mathbf{K} \mathbf{C}_\lambda + \mathbf{T} \mathbf{D}_\lambda. \quad (47)$$

The fact that this expression involves no differentiation means that $\mathbf{S}_\lambda$ can be easily computed on any scalp model. We also remark that the columns of $\mathbf{S}_\lambda$ sum to 1 (whereas the columns of $\mathbf{L}_\lambda$ sum to 0), implying that the transformation (46) preserves the reference potential.

There are two inconveniences to be highlighted about the parameter λ . First, the parameter λ has no upper bound, meaning that the search for the overall optimal value of λ should include any real number from 0 to infinity. In practice, this search is performed on a finite interval [Babiloni et al., 1995], but there is no guarantee of convergence to the overall optimal λ . Second, the optimal value of λ has no meaningful interpretation for comparison between different studies. For this reason, instead of thinking in terms of λ it is convenient to think in terms of an alternative parameter DF_λ that is called the *effective degree of freedom* of $f_\lambda(\mathbf{r})$. The parameter DF_λ is defined by [Hastie et al., 2009, James et al., 2013]

$$\text{DF}_\lambda = \text{trace}(\mathbf{S}_\lambda), \quad (48)$$

where the trace operation sums over the diagonal elements of $\mathbf{S}_\lambda$. The trace operation assigns a finite value to DF_λ between 1 ($\lambda = \infty$) and N ($\lambda = 0$), the lower bound²⁰ representing the linear portion of $f_\lambda(\mathbf{r})$ and the upper bound reflecting the maximum variance, for which $\mathbf{S}_\lambda$ is the projection matrix from regression over $(V_1, \dots, V_N)^T$.

In order to use DF_λ instead of λ we need to invert the relation (48). This inversion is possible but it can only be done numerically. Figure 6 depicts the result for a montage with 32 sensors. Although DF_λ is shown as if it

²⁰The lower bound $\text{DF}_\lambda = 1$ holds for spherical splines. For regular splines DF_λ is at least equal to 2, depending on the dimension of the space.

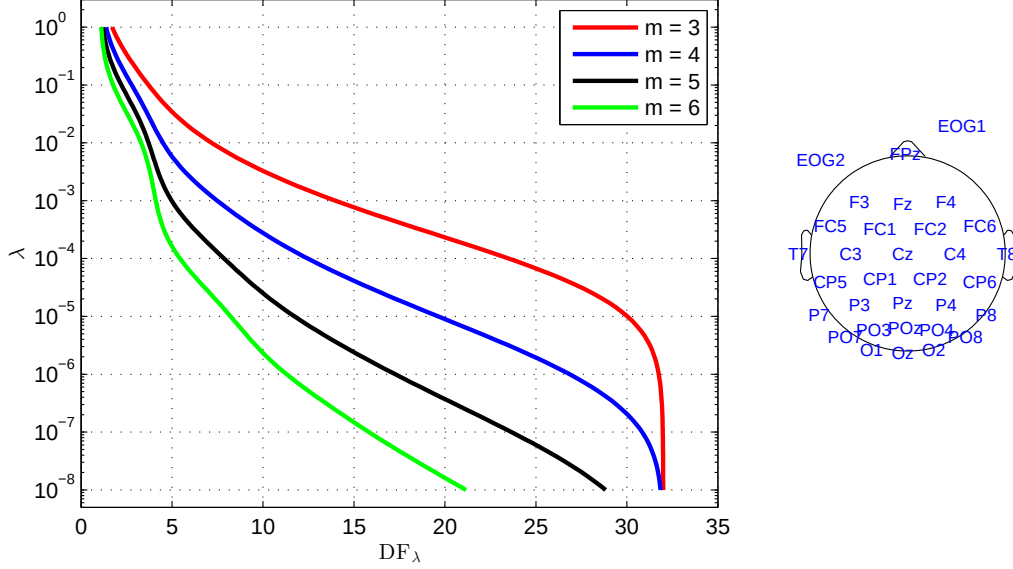


Figure 6: The monotonically decreasing relationship between DF_λ and λ for a montage with 32 electrodes (right). Each solid line indicates a distribution with 100 values of λ in the range of 10^{-8} to 1 on a log scale. The different colors indicate the value of the parameter m used in the calculation.

were the independent variable, its values were actually determined from (48) by fixing λ . The Matlab code of Appendix B was used to compute $\mathbf{S}_\lambda$. This code implements the spherical splines algorithm of Section 3.3 with a default error tolerance of less than 1×10^{-10} to evaluate the infinity series (32b). The electrode locations were normalized to a sphere of radius 10 cm. Figure 6 shows that DF_λ decreases monotonically with λ . Thanks to monotonicity we can interpolate the pairs (DF_λ, λ) to obtain a continuous curve from which we can estimate λ given DF_λ . To fix ideas an example using real brain data is discuss in detail in the supplementary material.

4. Concluding Remarks

The literature on the SL technique is quite large and this paper could only review or expand a few topics of general importance. It was our goal to give an intuitive view of the technique by providing physical insights that are often missing in the literature. In addition, we discussed some numerical methods to estimate the Laplacian, emphasizing each of its own set of strengths and limitations. Special attention was given to finite difference and spline methods. Finite difference is one of the simplest approaches for SL estimates, and its computational cost is advantageous, particularly for large number of electrodes. Its major disadvantage, however, is that estimates are prone to discretization errors and regularized estimates to eliminate spatial noise are not provided. Splines are still the main alternative to finite difference. The need for evenly-spaced electrodes and problems related to discretization are eliminated, the scalp geometry is more realistic, and spatial noise is can be reduced through regularization. The cost of such improvement is reflected in an increase in mathematical complexity and a higher computational cost compared to finite difference.

Splines are sensitive to error in electrode locations [e.g. Bortel and Sovka, 2008], a problem affecting finite difference as well. The belief that accurate estimates at peripheral electrodes are made possible by splines is not entirely correct. In fact, note that the construction of $f_\lambda(\mathbf{r})$ involves $N + M$ unknowns, which are the coefficients c_i 's and d_i 's in (24) and (32a), but only N observations are available for each frame, which are the measured potentials

$V_1, \dots, V_N$. To avoid underdetermination, the system (25) contains an ad hoc condition $\mathbf{T}^T \mathbf{c} = 0$ that adds M more constraint to $f_\lambda(\mathbf{r})$, which has the geometric effect of canceling terms from $f_\lambda(\mathbf{r})$ that grow faster than a polynomial of degree m as one moves away from the electrodes. Estimates at central and peripheral electrodes are affected differently by this condition, with estimates at peripheral electrodes being less accurate. It was outside the scope of our work to discuss more realistic scalp-model methods and electrode location error, as they require tools such as MRI that are often not readily available to researchers [Le and Gevins, 1993, Gevins et al., 1994, Le et al., 1994, Babiloni et al., 1996, Bortel and Sovka, 2013].

Our discussion recommended to set the spline parameter m as equal to 3 or 4 to avoid numerical issues. Figure 6 gives further insight into this problem. This example shows a compromise between m , the effective number of degrees of freedom of the spline fit, and the lower bound of λ . Although small and large values of m appear equally suitable to assess low degrees of freedom, say $\text{DF}_\lambda < 10$, the fact that λ is bounded from below at 10^{-8} prevents us from exploiting the region of $\text{DF}_\lambda > 20$ properly without setting $m = 3$ or 4. One may try to circumvent this limitation by decreasing the lower bound of λ toward zero, but as explained in Section 3.2 this has the problem of impoverishing the numerical conditioning, thus reinforcing our above recommendation.

Despite the broad literature on the subject, some basic problems on Laplacian estimates are still unsolved. One difficult problem that could not be

addressed is related to the finite size of electrodes. All computational methods discussed here were built on the idea of a point-like electrode, but an electrode has a finite size, and for this reason EEG potentials should be interpreted as spatial averages instead of point values. A possible approach to this consists of replacing the variational problem (23) with

$$\text{RSS}(f, \lambda) = \frac{1}{N} \sum_i \left[V_i - \iint_{\Omega_i} f(\mathbf{r}_i) dA \right]^2 + \lambda J_m[f], \quad (49)$$

where the double integral is over the area of the finite electrode. That is, we replace the interpolation condition $V_i = f(\mathbf{r}_i)$ by the *volume matching condition* $V_i = \iint_{\Omega_i} f(\mathbf{r}_i) dA$. Wahba [1981a] studied this problem in Euclidean space and used the GCV statistics to estimate the optimal value of the parameter λ . We seek a solution to the minimization problem on the sphere to deal with finite-sized electrodes on a spherical scalp model. This problem is arguably not trivial and almost certainly will require a numerical solution. Yet this is a simplified model that does not address the much more complicated problem of boundary effects due to the highly conductive electrode over the region Ω_i .

Considerable attention was devoted to the method of estimating the Laplacian derivation by means of a linear transformation, for which some figures were given estimating computational costs and encouraging the use of this procedure, mainly in the context of splines. Similar attention was paid to the problem of regularization to eliminate spatial noise. It is worth

remarking that spatial regularization is not useful to eliminate noise coming from the reference electrode. In fact, because spline fits preserves the reference signal, the residuals $\varepsilon_i = V_i - f_\lambda(\mathbf{r}_i)$ are reference-free, and so is the GCV error function, making evident that any noise due to the reference is fully preserved by regularization. This problem, however, should not cause much concern as it simply does not exist in the context of SL analysis.

Finally, we want to mention our effort in recent papers to develop a vector form of EEG that combines the SL and the tangential components of the electric field projected on the scalp surface [Wang et al., 2012, Carvalhaes et al., 2014]. This combination was grounded on the observation that the SL derivation is closely related to the spatial component of the electric field oriented normally to the scalp surface, thus not containing substantial information about features encoded in tangential direction. Not surprisingly, this combination of physically distinct but somewhat supplementary quantities resulted in significant improvement for a variety of classification tasks as described in Carvalhaes et al. [2014].

Appendix A. The Mathematics Behind Finite Difference

A systematic way to obtain finite difference approximations for the derivatives of a function is based on the *Taylor series expansion*. To expand in terms of Taylor series, we formulate the problem in terms of a continuous function V and then, for compatibility with our notation in Section 3.1, rewrite the final approximation using discrete variables. In its simplest case, the Taylor

series of a univariate function $V(x)$ around a point $x = a$ is defined by the infinite summation

$$V(a + h) = V(a) + V'(a)h + \frac{1}{2!}V''(a)h^2 + \frac{1}{3!}V'''(a)h^3 + \dots, \quad (\text{A-1})$$

where h is called the increment, $n! = n(n - 1)(n - 2) \cdots 1$ is the factorial of n , and $V^{(n)}(a)$ denotes the n -th derivative of $V(x)$ at $x = a$. That is, the Taylor series of $V(x)$ is a power series of h with coefficients given by the derivatives of $V(x)$ at $x = a$. To obtain an approximation for the second derivative (or any other derivative) of $V(x)$ at $x = a$ we proceed as follows. Replace h with $-h$ in (A-1) and obtain an analogous expression for $V(a - h)$, i.e.,

$$V(a - h) = V(a) - V'(a)h + \frac{1}{2!}V''(a)h^2 - \frac{1}{3!}V'''(a)h^3 + \dots$$

Adding these two expressions,

$$V(a + h) + V(a - h) = 2V(a) + V''(a)h^2 + \frac{1}{12}V^{iv}(a)h^4 + \dots,$$

whence

$$V''(a) = \frac{V(a + h) + V(a - h) - 2V(a)}{h^2} - \frac{1}{12}V^{iv}(a)h^2 - \dots, \quad (\text{A-2})$$

Assuming that h is sufficiently small we can ignore all terms containing powers of h and arrive at

$$V''(a) \approx \frac{V(a+h) + V(a-h) - 2V(a)}{h^2}.$$

Replacing $a = (i-1)h$, $i = 2, \dots, N-1$, and using the notation $V_i = V(a)$ and $V_{i\pm 1} = V(a \pm h)$ yields

$$V''(a) \approx \frac{V_{i+1} + V_{i-1} - 2V_i}{h^2}. \quad (\text{A-3})$$

This expression was obtained by truncating the infinite series (A-2) after the first term. This way, starting from h^2 all powers of h were eliminated assuming that h was sufficiently small. The result was a *second-order approximation*²¹, which in the “big-Oh” notation is referred to as $\mathcal{O}(h^2)$.

Approximation (A-3) excludes the case of a peripheral node, for which either V_{i+1} or V_{i-1} does not exist. In this case, we have to build the approximation using only nodes to the right or left of a , but not both. For instance, let $i = 1$ and use the expansions

$$\begin{aligned} V(a+h) &= V(a) + V'(a)h + \frac{1}{2}V''(a)h^2 + \frac{1}{6}V'''(a)h^3 + \dots, \\ V(a+2h) &= V(a) + 2V'(a)h + 2V''(a)h^2 + \frac{4}{3}V'''(a)h^3 + \dots, \end{aligned}$$

²¹The order of a finite difference approximation is given by the smallest power of h that is left out in the truncation.

from which

$$V(a + 2h) - 2V(a + h) = -V(a) + V''(a)h^2 + V'''(a)h^3 + \dots$$

and

$$V'''(a) = \frac{V(a + 2h) - 2V(a + h) + V(a)}{h^2} - V'''(a)h - \dots$$

In terms of the grid variables,

$$V''(a) \approx \frac{V_3 - 2V_2 + V_1}{h^2}, \quad (\text{A-4})$$

which is a *first order approximation*. Note that this expression corresponds to the formula obtained by Freeman and Nicholson [1975] for one-dimensional estimate of intracranial CSD.

A1. Bivariate functions and approximations for $\text{Lap}(V(a, b))$

The extension of (A-1) to bivariate functions replaces ordinary derivatives with partial derivatives. Namely,

$$\begin{aligned} V(a+h_1, b+h_2) = & V(a, b) + \left[\frac{\partial V(a, b)}{\partial x} h_1 + \frac{\partial V(a, b)}{\partial y} h_2 \right] + \frac{1}{2} \left[\frac{\partial^2 V(a, b)}{\partial x^2} h_1^2 + 2 \frac{\partial^2 V(a, b)}{\partial x \partial y} h_1 h_2 \right. \\ & \left. + \frac{\partial^2 V(a, b)}{\partial y^2} h_2^2 \right] + \frac{1}{6} \left[\frac{\partial^3 V(a, b)}{\partial x^3} h_1^3 + 3 \frac{\partial^3 V(a, b)}{\partial^2 x \partial y} h_1^2 h_2 + 3 \frac{\partial^3 V(a, b)}{\partial x \partial^2 y} h_1 h_2^2 + \frac{\partial^3 V(a, b)}{\partial y^3} h_2^3 \right] + \dots \end{aligned} \quad (\text{A-5})$$

Similar to the one-dimensional case, with appropriate choices for h_1 and h_2 one can use this expansion to approximate the partial derivatives of $V(x, y)$

at any point in a rectangular grid. An approximation for the Laplacian of $V(x, y)$ at a central node of a square grid ($h_1 = h_2 = h$) is obtained as follows.

First, use (A-5) to expand $V(a \pm h, b)$ and $V(a, b \pm h)$:

$$\begin{aligned} V(a \pm h, b) &= V(a, b) \pm \frac{\partial V(a, b)}{\partial x} h + \frac{1}{2} \frac{\partial^2 V(a, b)}{\partial x^2} h^2 \pm \frac{1}{6} \frac{\partial^3 V(a, b)}{\partial x^3} h^3 + \frac{1}{24} \frac{\partial^4 V(a, b)}{\partial x^4} h^4 \pm \dots, \\ V(a, b \pm h) &= V(a, b) \pm \frac{\partial V(a, b)}{\partial y} h + \frac{1}{2} \frac{\partial^2 V(a, b)}{\partial y^2} h^2 \pm \frac{1}{6} \frac{\partial^3 V(a, b)}{\partial y^3} h^3 + \frac{1}{24} \frac{\partial^4 V(a, b)}{\partial y^4} h^4 + \dots. \end{aligned}$$

Second, add these four expressions to obtain

$$\begin{aligned} V(a+h, b) + V(a-h, b) + V(a, b+h) + V(a, b-h) &= 4V(a, b) + \underbrace{\left[\frac{\partial^2 V(a, b)}{\partial x^2} + \frac{\partial^2 V(a, b)}{\partial y^2} \right]}_{=\text{Lap}(V(a, b))} h^2 \\ &\quad + \frac{1}{12} \left[\frac{\partial^4 V(a, b)}{\partial x^4} + \frac{\partial^4 V(a, b)}{\partial y^4} \right] h^4 + \dots, \end{aligned}$$

whence

$$\text{Lap}(V(a, b)) = \frac{V(a+h, b) + V(a-h, b) + V(a, b+h) + V(a, b-h) - 4V(a, b)}{h^2} + \mathcal{O}(h^2).$$

Therefore, $\text{Lap}(V(a, b))$ is approximated to the second-order of h by

$$\text{Lap}(V(a, b)) \approx \frac{V(a+h, b) + V(a-h, b) + V(a, b+h) + V(a, b-h) - 4V(a, b)}{h^2}.$$

This expression is equivalent to Hjorth's Laplacian estimate, which is shown in (19) with $V_{(i,j)} = V(a, b)$, $V_{(i \pm 1, j)} = V(a, b \mp h)$, and $V_{(i, j \pm 1)} = V(a \pm h, b)$.

A2. Approximations for peripheral electrodes

Suppose that we want to approximate $\text{Lap}(V)$ at a point (a, b) which is outside the domain of V . Let us start with a node in the left border of the grid, but not in a corner. Our mapping from the continuous variables x and y into the discrete variables i and j follows $x = (j - 1)h$ and $y = (N - i + 1)h$. See also the scheme in Figure 2. For a node on the left border $x = 0$, which corresponds to the first column of the grid ($j = 1$). The Taylor series of interest are

$$\begin{aligned} V(h, b) &= V(0, b) + \frac{\partial V(0, b)}{\partial x}h + \frac{1}{2}\frac{\partial^2 V(0, b)}{\partial x^2}h^2 + \frac{1}{6}\frac{\partial^3 V(0, b)}{\partial x^3}h^3 + \frac{1}{24}\frac{\partial^4 V(0, b)}{\partial x^4}h^4 + \dots, \\ V(2h, b) &= V(0, b) + 2\frac{\partial V(0, b)}{\partial x}h + 2\frac{\partial^2 V(0, b)}{\partial x^2}h^2 + \frac{4}{3}\frac{\partial^3 V(0, b)}{\partial x^3}h^3 + \frac{2}{3}\frac{\partial^4 V(0, b)}{\partial x^4}h^4 + \dots, \\ V(0, b \pm h) &= V(0, b) \pm \frac{\partial V(0, b)}{\partial y}h + \frac{1}{2}\frac{\partial^2 V(0, b)}{\partial y^2}h^2 \pm \frac{1}{6}\frac{\partial^3 V(0, b)}{\partial y^3}h^3 + \frac{1}{24}\frac{\partial^4 V(0, b)}{\partial y^4}h^4 \pm \dots. \end{aligned}$$

A linear combination of these expansions with weights $-2, 1, 1$, and 1 gives

$$\begin{aligned} -2V(h, b) + V(2h, b) + V(0, b+h) + V(0, b-h) &= V(0, b) + \underbrace{\left[\frac{\partial^2 V(0, b)}{\partial x^2} + \frac{\partial^2 V(0, b)}{\partial y^2} \right]}_{=\text{Lap}(V(0, b))} h^2 \\ &\quad + \frac{\partial^3 V(0, b)}{\partial x^3} h^3 + \dots, \end{aligned}$$

so that

$$\text{Lap}(V(0, b)) = \frac{-2V(h, b) + V(2h, b) + V(0, b+h) + V(0, b-h) - V(0, b)}{h^2} - \frac{\partial^3 V(0, b)}{\partial x^3} h.$$

Assuming that h is sufficiently small we obtain

$$\text{Lap}(V(0, b)) \approx \frac{V(2h, b) - 2V(h, b) + V(0, b + h) + V(0, b - h) - V(0, b)}{h^2},$$

which is a *first-order approximation*. In terms of the grid variables,

$$\text{Lap}(V)|_{(i,1)} \approx \frac{V_{(i,3)} - 2V_{(i,2)} + V_{(i-1,1)} + V_{(i+1,1)} - V_{(i,1)}}{h^2}, \quad (\text{A-8a})$$

In a similar fashion, for grid points at right, bottom, and upper edges (excluding the corners) we obtain, respectively,

$$\text{Lap}(V)|_{(i,N)} \approx \frac{V_{(i,N-2)} - 2V_{(i,N-1)} + V_{(i-1,N)} + V_{(i+1,N)} - V_{(i,N)}}{h^2}, \quad (\text{A-8b})$$

$$\text{Lap}(V)|_{(N,j)} \approx \frac{V_{(N-2,j)} - 2V_{(N-1,j)} + V_{(N,j-1)} + V_{(N,j+1)} - V_{(N,j)}}{h^2} \quad (\text{A-8c})$$

$$\text{Lap}(V)|_{(1,j)} \approx \frac{V_{(3,j)} - 2V_{(2,j)} + V_{(1,j-1)} + V_{(1,j+1)} - V_{(1,j)}}{h^2} \quad (\text{A-8d})$$

Finally, assume that the node of interest is at the left upper corner of the grid, corresponding to $i = j = 1$ ($a = 0$, $b = (N - 1)h$). To approximate the Laplacian of V at this node we use

$$\begin{aligned} V(h, b) &= V(0, b) + \frac{\partial V(0, b)}{\partial x}h + \frac{1}{2}\frac{\partial^2 V(0, b)}{\partial x^2}h^2 + \frac{1}{6}\frac{\partial^3 V(0, b)}{\partial x^3}h^3 + \frac{1}{24}\frac{\partial^4 V(0, b)}{\partial x^4}h^4 + \dots, \\ V(2h, b) &= V(0, b) + 2\frac{\partial V(0, b)}{\partial x}h + 2\frac{\partial^2 V(0, b)}{\partial x^2}h^2 + \frac{4}{3}\frac{\partial^3 V(0, b)}{\partial x^3}h^3 + \frac{2}{3}\frac{\partial^4 V(0, b)}{\partial x^4}h^4 + \dots, \\ V(0, b - 2h) &= V(0, b) - 2\frac{\partial V(0, b)}{\partial y}h + 2\frac{\partial^2 V(0, b)}{\partial y^2}h^2 - \frac{2}{3}\frac{\partial^3 V(0, b)}{\partial y^3}h^3 + \frac{2}{3}\frac{\partial^4 V(0, b)}{\partial y^4}h^4 + \dots, \\ V(0, b - h) &= V(0, b) - \frac{\partial V(0, b)}{\partial y}h + \frac{1}{2}\frac{\partial^2 V(0, b)}{\partial y^2}h^2 - \frac{1}{6}\frac{\partial^3 V(0, b)}{\partial y^3}h^3 + \frac{1}{24}\frac{\partial^4 V(0, b)}{\partial y^4}h^4 + \dots. \end{aligned}$$

Adding these expansions with weights -2 , 1 , 1 , and -2 yields

$$\begin{aligned} -2V(h, b) + V(2h, b) + V(0, b-2h) - 2V(0, b-h) = & -2V(0, b) + \underbrace{\left[\frac{\partial^2 V(0, b)}{\partial x^2} + \frac{\partial^2 V(0, b)}{\partial y^2} \right]}_{=\text{Lap}(V(a, b))} h^2 \\ & + \left[\frac{\partial^3 V(0, b)}{\partial x^3} - \frac{\partial^3 V(0, b)}{\partial y^3} \right] h^3 + \dots \end{aligned}$$

Solving for $\text{Lap}(V)$ we obtain the first-order approximation

$$\text{Lap}(V(0, b)) \approx \frac{V(2h, b) - 2V(h, b) + V(0, b-2h) - 2V(0, b-h) + 2V(0, b)}{h^2}.$$

Equivalently,

$$\text{Lap}(V)|_{(1,1)} \approx \frac{V_{(1,3)} - 2V_{(1,2)} + V_{(3,1)} - 2V_{(2,1)} + 2V_{(1,1)}}{h^2}. \quad (\text{A-10})$$

Using symmetry we can obtain similar approximations for the other corners of the grid illustrated in Figure 3.

Appendix B. A Matlab Code to Construct $\mathbf{S}_\lambda$ and $\mathbf{L}_\lambda$

Here we present a Matlab code to build the smoother matrix $\mathbf{S}_\lambda$ and the SL matrix $\mathbf{L}_\lambda$ using spherical splines. The first function, `sphlap0.m`, is an auxiliary function to generate the matrices $\mathbf{K}$, $\tilde{\mathbf{K}}$, $\mathbf{T}$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{R}$ presented above. Since none of these matrices depend on the regularization parameter λ , they need to be computed only once for each value of m and electrode configuration. The computation of these matrices is not fast

because it involves the evaluation of Legendre polynomials. Therefore, for efficiency, this function should never be called inside a regularization loop. The second function, `sphlap.m`, receives the output of `sphlap0.m` as argument plus the value of λ and return the matrices $\mathbf{S}_\lambda$ and $\mathbf{L}_\lambda$. This is the function that should be used inside a regularization loop.

Listing 1: Auxiliar function to compute spherical splines

```
% sphlap0.m - Auxiliar function to compute spherical splines
%
% Usage: [K, LapK, T, Q1, Q2, R] = sphlap0(x, y, z, m, tol);
%
% Required Inputs
% x, y, z: electrode coordinates (must be on a sphere)
% m: interpolation order (2<=m<=6)
% Optional Input
% tol: error tolerance in the estimate of the Legendre polynomials.
% Default is 1e-10.
%
% Output
% K, LapK, T, Q1, Q2, R: matrices required to implement spherical splines
%

function [K, LapK, T, Q1, Q2, R, tol] = sphlap0(x, y, z, m, tol)

% Handle arguments
if nargin < 4
    help sphlap0.m;
    return;
end

if nargin == 4
    tol = 1e-10;
end

if m < 2 || m > 6
    error('sphlap0:intorder','The parameter "m" should be in the range of
        %s',...
        '2 and 6');
end

r = hypot(z, hypot(x, y)); % head radius
N = length(x); % number of electrodes

sqdist = pdist([x(:) y(:) z(:)], 'euclidean').^2; % square distances
        between electrodes
dr2 = squareform(sqdist); % convert distances to matrix form
```

```

r2 = r(1)^2; % squared head radius
cos_gamma = 1 - dr2/(2*r2); % angle between electrodes

if any((cos_gamma(:) > 1) | (cos_gamma(:) < -1))
    error('sphlap0:locs', ...
        'Something is wrong with the electrode coordinates. %s', ...
        'Are they on located on a sphere');
end

G = [];
LapG = [];
G0 = 0;
epsilon = tol + 1;
n = 1;
while (tol < epsilon)
    Pn = legendre(n, cos_gamma(:));
    a = (2*n+1) / (n*(n+1))^m;
    gm = a * Pn(1,:);
    G = horzcat(G, gm);
    LapG = horzcat(LapG, -n * (n+1) * gm);
    epsilon = max(abs(G(:,end)-G0));
    G0 = G(:,end);
    n = n + 1;
end

tol = epsilon; % final error tolerance

K = reshape(sum(G,2), N, [])/(4*pi);
LapK = reshape(sum(LapG,2), N, [])/(4*pi*r2);
T = ones(N,1);

% QR decomposition of T
[Q, R] = qr(T);
R = R(1);
Q1 = Q(:,1);
Q2 = Q(:,2:N);

% Alternative
% R = -sqrt(N);
% Q1 = T / R;
% [U,~,~] = svd(T);
% Q2 = U(:,2:end);

end

```

Listing 2: Function to compute the smoother and the Laplacian matrices

```

% sphlap.m - Compute the smoother and Laplacian matrices using spherical
%           splines
%
% Usage: [S, L] = sphlap(K, LapK, T, Q1, Q2, R, lambda);
%

```

```

% Required Inputs
% K, LapK, T, Q1, Q2, and R: matrices generated by sphlap0
% lambda: smoothing parameter
%
% Output:
% S = the smoother matrix
% L = the Laplacian matrix
%

function [S, L] = sphlap(K, LapK, T, Q1, Q2, R, lambda)

% Handle arguments
if nargin ~= 7
    help sphlap.m;
    return;
end

I = eye(size(K,1));
Klamb = K + lambda*I;

C = Q2 / (Q2' * Klamb * Q2) * Q2';
D = R \ Q1' * (I - Klamb*C);

S = K * C + T * D; % The smoother matrix
L = LapK * C; % The Laplacian matrix

end

```

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