

Negative probabilities and counter-factual reasoning in quantum cognition

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Abstract. In this paper we discuss quantum-like decision-making experiments using negative probabilities. We do so by showing how the two-slit experiment, in the simplified version of the Mach-Zehnder interferometer, can be described by this formalism. We show that negative probabilities impose constraints to what types of counter-factual reasoning we can make with respect to (quantum) internal representations of the decision maker.

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1. Introduction

The use of the mathematical apparatus of quantum mechanics in the social sciences has experienced a boom in recent years, with whole books solely devoted to certain aspects of it (see [1, 2, 3] and references therein). Such an endeavour has been dubbed “quantum interactions,” and it has been the topic of many academic meetings, such as the special section in Andrei Khrennikov’s foundations of quantum mechanics conference. The goal of the quantum interaction community is not to search for “true” quantum mechanical effects, but instead to ask whether there are social phenomena that could be better described by, say, vectors on a Hilbert space and Hermitian operators representing observable quantities.

In many cases the quantum-like probabilistic structure is modelled by introducing the use of “quantum” intermediate states, often not accessible under certain experimental conditions, whose superposition are similar to the superposition of the two quantum waves, ψ_A and ψ_B , in analogy to the two-slit experiment. This is, for instance, how the violation of Savage’s Sure-Thing Principle (STP) is modelled with state vectors [4]. Thus, it should come as no surprise that one of the often used analogies is interference in the two-slit experiment, since it is perhaps best illustrates some of the puzzling quantum aspects [1, 2, 3, 5, 6, 7].

The importance of the two slit in physics is emphasized by Richard Feynman in his celebrated lectures on physics [8]. There, Feynman states that the *only* mystery in quantum mechanics is contained in this experiment, and that it “is impossible, absolutely impossible, to explain in any classical way” this phenomenon. Though grossly inaccurate[‡], Feynman’s claim carries a certain element of truth, as the outcomes of the two slit experiment seem to make no sense if we think solely in terms of particles and detectors. At the heart of such mystery is connection between the particle’s trajectory and its surroundings (the context), yielding probabilistic outcomes that are not only context-dependent but also non-monotonic.

The above “mystery” raises the prospect of thinking of quantum-like effects in the social sciences as originated by the interference effects of highly contextual variables [10, 11]. For instance, in the case of human decision making, one can show that, under reasonable assumptions, neural oscillators in the brain may lead to interference-like effects that are similar to a two-slit set-up [12, 13, 14]. This interference is highly contextual, as it depends on boundary conditions on the dynamics of neural oscillators, and can not be described by standard probability theory. Furthermore, such contextual variables may have a description in terms of extended probabilities that is more general than the standard quantum formalism [15, 16, 17].

Part of the mystery is related to the use of counter-factual reasoning [18, 19]. Counter-factual reasoning in quantum mechanics is based on non-measured properties of a system. Suppose that we have two observations $\mathbf{A} = 1$ and $\mathbf{B} = 1$, with $\mathbf{A}$

[‡] For example, as Feynman was well aware, David Bohm provided a way to think about the two slit in terms of classical concepts of waves and particles [9].

preceding (perhaps even causally) $\mathbf{B}$. In this case, a counter-factual reasoning would be something like “if $\mathbf{A} = 1$ had not been observed, $\mathbf{B} = 1$ would not have been observed.” It is often pointed out that the inconsistencies associated with quantum mechanics come from the use of counter-factual reasoning. For example, in the famous GHZ paradox [20], inconsistencies are derived from the assumption that values of some observables are defined even if experimental conditions when they had not been observed, which is equivalent to requiring the existence of a non-negative joint probability distribution [21].

The use of negative probabilities connected to the quantum formalism dates back from the seminal works of Eugene Wigner [22], where he showed that if one wanted a joint probability distribution of momentum and position that would reproduce quantum statistical mechanics predictions, such joint would have to take negative values. Later on, Paul Dirac [23] proposed the use of negative probabilities to understand the quantization of electromagnetic fields. However, as Richard Feynman remarked, even though negative probabilities seem to be a part of quantum mechanics, no practical use was ever found for them [24], although recently some authors proposed possible applications [17, 25, 26].

Perhaps one of the difficulties of using negative probabilities is its unclear physical meaning. Wigner considered them nonsensical, and both Dirac and Feynman thought they were mere computational devices devoid of meaning. However, recently many different interpretations have appeared. For example, in [16] a possible subjective interpretation in terms of non-monotonic upper probability distributions was proposed, and in [25] a map based on positive and negative measures gave a possible frequentist interpretation to negative probabilities. Another approach was that of Khrennikov, who showed that negative probabilities are obtained in the context of p -adic statistics, and are the consequence of a violation of principle of statistical stabilization [27, 28, 29, 30, 31]. Be that as it may, negative probabilities in quantum mechanics seem to be related to the appearance of correlations that are too strong to be explained by a context-independent hidden-variable theory [32], such as the case for the well-known Bell-EPR systems.

Our goal in this paper is to examine the use of extended probabilities in the context of the two-slit experiment, as suitable for discussions in quantum cognition. In particular, we examine this experiment in its simplified version: the Mach-Zehnder interferometer. We then discuss what constraints are imposed to a rational description of the system when further (counter-factual) information is assumed about the system. To do so, we organize this paper in the following way. In Section 2 we discuss a typical application of the quantum formalism in decision making, showing its equivalence to the quantum Mach-Zehnder interferometer. In Section 3 we analyse in terms of negative probabilities the Mach-Zehnder set-up, and show that certain types of counter-factual reasoning are inconsistent with them.

2. Interference in Quantum Cognition

Let us start with the famous violation of STP, observed empirically by Tversky and Shafir [33, 34]. In one of their experiments, participants (students in a college-level class) were asked whether they would buy a certain ticket under three different contexts. In the first context, they were asked if they would buy if they passed the class. In the second context, they were asked if they would buy if they failed the class. Finally, in the third context they were asked whether they would buy the ticket if they didn't know whether they passed the class or not. Participants consistently preferred to buy the tickets when they knew whether they passed or failed the class than if they didn't know, a violation of STP.

To understand this violation of STP, it helps to write the situation in a more formal way. Let $\mathbf{P}$ and $\mathbf{B}$ be two ± 1 -valued random variables. $\mathbf{B}$ represents the intent to buy a ticket, and $\mathbf{P}$ represents passing the class, with both taking value 1 if the answer was “yes” and -1 if “no”. The probability of purchasing a ticket in the first context corresponds to the value $P(b|p)$, the second context to $P(b|\bar{p})$. Since in the third context the participant does not know, but 1 and -1 are the only possible outcomes, the third context must be $P(b|p \cup \bar{p}) = P(b)$. Here we use the standard notation that b represents $\mathbf{B} = 1$, $\bar{b}$ represents $\mathbf{B} = -1$, and so on. Since the probability of $\mathbf{B} = 1$ is given by

$$P(b|p \cup \bar{p}) = P(b) = P(b|p)P(p) + P(b|\bar{p})P(\bar{p}), \quad (1)$$

with $P(p) + P(\bar{p}) = 1$, it follows that $P(b|p \cup \bar{p})$ is a convex combination of $P(b|p)$ and $P(b|\bar{p})$. Thus, $P(b|p \cup \bar{p}) \geq P(b|p)$. This, of course, contradicts the results of Tversky and Shafir, and shows that their participants violate STP, a consequence of the standard theory of probabilities.

It is worthwhile mentioning here that the main characteristic of the data, as the one discussed above, is to show that, under certain circumstances, human decision making is non-monotonic. Recall that standard probability theory is monotonic, in the sense that for two sets A and B with $B \subseteq A$, $P(A) \geq P(B)$. Monotonicity is a consequence of the additivity rule, which gives $P(A) = P(B \cup \bar{B}) = P(B) + P(\bar{B})$, with $\bar{B} = A \setminus B$ being the relative complement of B in A , plus the non-negativity axiom. This means that certain extended probability theories that are monotonic, such as Dempster-Shafer [35, 36], may not be adequate to describe actual human decision making.

Non-monotonicity is directly connected to the quantum formalism via quantum interference. To understand this, let us examine the Mach-Zehnder interferometer, depicted in Figure 1, as a simplified two-slit experiment. The input into the system is the state $|1\rangle_S|0\rangle_V$, corresponding to one photon in channel S and no photon in the vacuum channel V . After the first beam splitter the state becomes

$$\frac{1}{\sqrt{2}} [|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B],$$

where the subscripts A and B are the modes in the arms of the interferometer, and the state corresponds to a superposition of a photon in A or B . Ignoring the reflection on

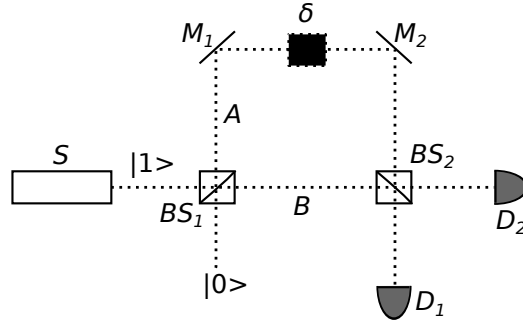


Figure 1. Mach-Zehnder interferometer. A single particle emitted from source S reaches beam-splitter BS_1 , and has a 50/50 probability of going to either arm, A or B , of the interferometer. A second beam-splitter BS_2 then joints both paths A and B and sends them to detectors D_1 and D_2 . The intensities seen on D_1 and D_2 depend on the relative phases of each beam, determined by the phase shifter δ .

the mirrors, since what is relevant is the phase introduced by δ , the state impinging upon the second beam splitter is

$$\frac{1}{\sqrt{2}} [e^{i\delta} |1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B].$$

Finally, the state impinging upon the detectors is given by

$$\frac{1}{2} (1 - e^{i\delta}) |1\rangle_1 |0\rangle_2 + \frac{1}{2} (1 + e^{i\delta}) |0\rangle_1 |1\rangle_2,$$

where subscripts 1 and 2 refer to detectors D_1 and D_2 respectively. As we can see, the probability of detections are $P(d_1) = \frac{1}{2} (1 - \cos \delta)$ and $P(d_2) = \frac{1}{2} (1 + \cos \delta)$, and if we select $\delta = \pi$, the probability of detection on D_1 is one whereas on D_2 it is zero. However, if we measure the particle in one of the interferometer arms, say after the phase δ is introduced, we end up with a mixture of two possible states, $|1\rangle_a |0\rangle_b$ or $|0\rangle_a |1\rangle_b$. After the last beam splitter, we get the state

$$\frac{1}{\sqrt{2}} [|1\rangle_1 |0\rangle_2 + |0\rangle_1 |1\rangle_2],$$

and we see that the probabilities for detection are now $P(d_1) = P(d_2) = \frac{1}{2}$.

From the above computations we can now see the non-monotonicity of this set-up. Let us imagine two possibilities. Possibility one is such that the photon can only travel through path A , perhaps by simply placing a barrier on path B . Possibility two is when no barriers are placed, and the photon can freely travel through either arm of the interferometer. According to monotonic reasoning, the more possibilities we have going from S to D_2 , the higher the probability of D_2 . However, if we choose $\delta = \pi$, $P(d_2) = 0$ for both paths being available, and $P(d_2) = \frac{1}{2}$ for only one path available. In other words, $P(d_2|a \cup b) \leq P(d_2|a)$, similar to the violation of Savage's STP mentioned above.

To model the violation of Savage's STP with a quantum formalism, all we need to do is to think of the path information $|1\rangle_a$ or $|1\rangle_b$ as the “pass” or “fail” class intermediate states, say $|\text{pass}\rangle$ and $|\text{fail}\rangle$. The final detectors D_1 and D_2 corresponding to “buy” or

“not buy” the ticket. So, in one experimental condition, when the experimenter asks the answer if the participant passed the class, this would correspond to measuring a which path the particle travelled, thus collapsing the wave function in the $|\text{pass}\rangle$ state and destroying the interference. In other words, in the version of the experiment where participants are asked their answers if they passed or failed, their state evolves as

$$|\text{pass}\rangle \text{ or } |\text{fail}\rangle \longrightarrow \frac{1}{\sqrt{2}} [|\text{purchase}\rangle + |\text{not purchase}\rangle],$$

whereas in the other version the evolution is

$$\frac{1}{\sqrt{2}} [|\text{pass}\rangle + |\text{fail}\rangle] \longrightarrow \frac{1}{2} [(1 - e^{i\delta}) |\text{purchase}\rangle + (1 + e^{i\delta}) |\text{not purchase}\rangle],$$

where δ is a phase parameter responsible for interference.

3. Counter-factuals and Signed Probabilities

The violation of Kolmogorov’s axioms can be shown to correspond, in the case of quantum systems, to the existence of a signed probability distribution [26, 17, 25], where the non-negativity axiom is relaxed. As mentioned in the introduction, negative probabilities are the consequence of correlations between different random variables that cannot be explained by a context-independent hidden variable [32]. Though the correlations themselves are observable in physical systems, negative probability states are not observable in principle [17], whereas this may not necessarily be true for social systems where negative probabilities appear [16, 37]. In this section we show how negative probabilities can be used in the Mach-Zehnder case shown above, and what are the possible insights that we may gain from it.

Let us imagine that we perform a destructive measurement. The constraints given by the marginals are

$$P(a\bar{b}) = P(\bar{a}b) = \frac{1}{2}, \quad (2)$$

and

$$P(ab) = P(\bar{a}\bar{b}) = 0, \quad (3)$$

which corresponds to having only one photon at a time. Here we use the notation a as corresponding a detection in the interferometer arm A , and $\bar{a}$ for no detection, and similarly for b , d_1 , and d_2 . A different experiment, when no detection is made for A or B , thus yielding interference at the detectors, requires

$$P(d_1\bar{d}_2) = 1 - \alpha, \quad (4)$$

$$P(\bar{d}_1d_2) = \alpha, \quad (5)$$

$$P(d_1d_2) = P(\bar{d}_1\bar{d}_2) = 0, \quad (6)$$

where $\alpha = 0$ or $\alpha = 1$ give maximum interferometric visibility. A non-negative probability distributions exists for (2)–(6) if we assume that (2) and (3) still holds

when we have interference. For example, a possible (non-unique) solution consistent with the marginals is given by

$$P(a\bar{b}d_1\bar{d}_2) = P(\bar{a}bd_1\bar{d}_2) = \frac{1}{2}(1 - \alpha),$$

$$P(a\bar{b}\bar{d}_1d_2) = P(\bar{a}b\bar{d}_1d_2) = \frac{1}{2}\alpha,$$

and the other atomic elements of the algebra (e.g., abd_1d_2 , $abd_1\bar{d}_2$, etc.) having probability zero.

But we also know, from another experimental set-up where we place a detector in *only one* of the interferometer arms, that when the particle is *not* in A and no detector in B , there is a probability of detection in D_1 and D_2 given by

$$P(\bar{a}d_1\bar{d}_2) = P(\bar{a}d_1d_2) = P(\bar{b}d_1\bar{d}_2) = P(\bar{b}d_1d_2) = \frac{1}{4}, \quad (7)$$

which implies the destruction of interference by which-path information. Contrary to the marginals (2)–(6), there is no non-negative joint probability distribution consistent with (4)–(7). However, if we relax the requirement that probabilities must be non-negative, it is possible to show that there is an infinite number of (negative) joint probability distributions that are consistent with (4)–(7).

Once the non-negativity requirement is relaxed, solutions consistent with marginals can start to have very large (negative and positive) values. To further constraint the negative probability solutions, we add the principle that the probability mass M , defined as $M = \sum |p_i|$ (p_i being the probability for each atom), should be minimized. Intuitively minimizing the $L1$ norm of the probability P corresponds to requiring that P be as close as possible to a non-negative probability distribution [16], and prevents solutions with large negative or positive values.

The general solution consistent with (4)–(7) and minimizing the probability mass M is, for the non-zero probability terms,

$$\begin{aligned} P(abd_1\bar{d}_2) &= -\frac{3}{4} + x + \alpha, & P(ab\bar{d}_1d_2) &= \frac{3}{4} - x - \alpha, \\ P(a\bar{b}d_1\bar{d}_2) &= \frac{1}{2} - x, & P(a\bar{b}\bar{d}_1d_2) &= x, \\ P(\bar{a}bd_1\bar{d}_2) &= \frac{1}{2} - x, & P(\bar{a}b\bar{d}_1d_2) &= x, \\ P(\bar{a}\bar{b}d_1\bar{d}_2) &= -\frac{1}{4} + x, & P(\bar{a}\bar{b}d_1d_2) &= \frac{1}{4} - x. \end{aligned} \quad (8)$$

M is minimized in (8) when $1/4 \leq x \leq 1/2$ for $0 \leq \alpha \leq 1/4$, $1/4 \leq x \leq 3/4 - \alpha$ for $1/4 \leq \alpha \leq 1/2$, $3/4 - \alpha \leq x \leq 1/4$ for $1/2 \leq \alpha \leq 3/4$, and $1/2 \leq x \leq 1/4$ for $3/4 \leq \alpha \leq 1$. Notice that if $\alpha = 1/2$, for no interference, the joint becomes

$$\begin{aligned} P(abd_1\bar{d}_2) &= P(ab\bar{d}_1d_2) = 0, \\ P(a\bar{b}d_1\bar{d}_2) &= P(a\bar{b}d_1d_2) = P(\bar{a}bd_1\bar{d}_2) = \\ P(\bar{a}\bar{b}d_1\bar{d}_2) &= P(\bar{a}\bar{b}d_1d_2) = P(\bar{a}\bar{b}\bar{d}_1d_2) = \frac{1}{4}, \end{aligned}$$

which is non-negative, as expected. However, for $\alpha \neq 1/2$, which is the case for interference, no non-negative solutions exist. For example, in the case of maximum

visibility, when $\alpha = 1$, a possible negative joint probability minimizing the probability mass M is given by (choosing $x = 1/4$)

$$\begin{aligned} P(abd_1\bar{d}_2) &= \frac{1}{2}, & P(ab\bar{d}_1d_2) &= -\frac{1}{2}, \\ P(a\bar{b}d_1\bar{d}_2) &= \frac{1}{4}, & P(a\bar{b}\bar{d}_1d_2) &= \frac{1}{4}, \\ P(\bar{a}bd_1\bar{d}_2) &= \frac{1}{4}, & P(\bar{a}b\bar{d}_1d_2) &= \frac{1}{4}, \\ P(\bar{a}\bar{b}d_1\bar{d}_2) &= 0, & P(\bar{a}\bar{b}\bar{d}_1d_2) &= 0. \end{aligned}$$

On the other hand, if we use counter-factual reasoning for the above probabilities, we have that if there is no detection in A , then we must have $\mathbf{B} = 1$, whereas no detection in B means $\mathbf{A} = 1$. Thus, the probabilities are

$$P(\bar{a}bd_1\bar{d}_2) = P(\bar{a}b\bar{d}_1d_2) = P(a\bar{b}d_1\bar{d}_2) = P(a\bar{b}\bar{d}_1d_2) = \frac{1}{4}, \quad (9)$$

and all other probabilities are zero. It is easy to show that no negative probabilities exist consistent with (4)–(6) and (9).

4. Final Remarks

We saw that in the two slit experiment, in the simpler context of the Mach-Zehnder interferometer, proper probability distributions exist if we do not make any *assumptions* about which-path information. However, once we include which-path information in the form of counter-factual reasoning, no proper joint exists, and negative probabilities are needed. Furthermore, if we assume more about the trajectory of the particle than what is contained in (7), i.e. if we include more counter-factual assumptions as in (9), not even a negative probability distribution exists.

The non-existence of a negative probability, as showed in [17], is related to the violation of the no-signalling condition, which in psychology is equivalent to the violation of marginal selectivity [38, 39]. This should not come as a surprise, because a Mach-Zehnder can be used to signal. To see this, imagine that Alice is at the arms of the interferometer, and Bob is looking at the detectors D_1 and D_2 . Alice's decision to measure or not where the particle is would affect the outcomes of Bob in a way that Bob could tell whether Alice is measuring or not. Of course, this type of signalling does not present any conceptual problems, as it is purely classical, since it would take some time for the particle to arrive from Alice and Bob, and their measurements would not be space-like separated.

Our results suggest that in quantum-like decision-making, where most effects come from cases where the superposition of two states result in quantum-like interference, we should expect violation of marginal selectivity (or the no-signalling condition in physics). Furthermore, if we want to use the empirical results to think about inaccessible internal reasoning states (e.g., if the student answered “buy a ticket”, was she thinking “passed class”?), since those states are not directly measured, we need to use counter-factual reasoning. However, as we showed above, not every counter-factual reasoning allows us to obtain information about the system, as a (negative) joint probability cannot always be found (such as in equations (9)). Thus, to think about those internal reasoning

states, one needs to restrict the counter-factual assumptions one makes or perhaps use a more general approach, such as the one proposed by Dzhafarov and Kujala [38].

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