

Probabilistic Results for Six Detectors in a Three-Particle GHZ Experiment

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Abstract

In this paper we show that the GHZ theorem can be reformulated as a probabilistic theorem allowing for inefficiencies in the detectors. We show quantitatively that taking into account these inefficiencies, the published results of the Innsbruck experiment support the nonexistence of a joint probability distribution for the six correlated spin variables, and hence the nonexistence of hidden variables that explain the experimental results.

1 Introduction

One of the most influential papers on the foundations of quantum mechanics is undoubtedly that of Einstein, Podolski and Rosen (EPR), where they analyzed the correlations between observables of entangled two-particle states [1]. In their paper, EPR conclude that, because entangled systems can be perfectly correlated, Bohr's interpretation of quantum mechanics was incomplete, and a new theory using hidden variables was necessary. However, in 1963 John Bell showed that if we wanted a hidden-variable theory for quantum mechanics, as EPR did, this theory would be falsifiable, as it would yield correlations incompatible with the ones predicted by quantum mechanics [2]. In 1982, in a famous experiment, Aspect, Dalibard and Roger showed that quantum mechanical correlations were satisfied, in contradiction to the local hidden-variable predictions [3].

Recently, Greenberger, Horne and Zeilinger (GHZ) proposed another test for quantum mechanics against hidden-variable theories [4]. What makes the GHZ proposal distinct from Bell's inequalities is the fact that, by using the quantum mechanical predicted correlations for entangled states of more than two particles, they could obtain a mathematical contradiction if a local hidden-variable theory were assumed. The contradiction obtained for the case of three particles, as derived by Mermin in [5], goes as follows. Let $|\psi\rangle$ be a three-particle entangled spin state defined by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle), \quad (1)$$

where we use the notation $|++\rangle = |+\rangle_1 \otimes |+\rangle_2 \otimes |+\rangle_3$, and $\hat{\sigma}_{1z}|+\rangle_1 = |+\rangle_1$, $\hat{\sigma}_{1z}|-\rangle_1 = -|-\rangle_1$, $\hat{\sigma}_{1z}$ being the spin observable in the z direction acting on particle 1, and so on. The wavefunction (1) is an eigenstate of the following operators:

$$\hat{A}|\psi\rangle = \hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}|\psi\rangle = |\psi\rangle, \quad \hat{B}|\psi\rangle = \hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}|\psi\rangle = |\psi\rangle, \quad (2)$$

$$\hat{C}|\psi\rangle = \hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}|\psi\rangle = |\psi\rangle, \quad \hat{D}|\psi\rangle = \hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}|\psi\rangle = -|\psi\rangle. \quad (3)$$

From equations (2)–(3) we have at once that

$$E(\hat{A}) = E(\hat{B}) = E(\hat{C}) = 1 \quad (4)$$

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and

$$E(\hat{D}) = -1. \quad (5)$$

To show the contradiction, we assume there is a hidden variable λ , that is a function of the source that generated the entangled state and that determines the value of each of the components of spin for the particles. Let us denote by $m_{ix}(\lambda)$ the value ± 1 of the spin $\hat{\sigma}_{ix}$ in the x direction for particle i , $i = 1, 2, 3$, and by $m_{iy}(\lambda)$ the value of the spin $\hat{\sigma}_{iy}$ in the y direction. Since the expectations are 1, we can write, from (2)–(3), that

$$m_{1x}(\lambda)m_{2y}(\lambda)m_{3y}(\lambda) = 1, \quad (6)$$

$$m_{1y}(\lambda)m_{2x}(\lambda)m_{3y}(\lambda) = 1, \quad (7)$$

$$m_{1y}(\lambda)m_{2y}(\lambda)m_{3x}(\lambda) = 1. \quad (8)$$

But

$$[m_{1x}m_{2y}m_{3y}][m_{1y}m_{2x}m_{3y}][m_{1y}m_{2y}m_{3x}] = [m_{1x}m_{2x}m_{3x}][m_{1y}^2m_{2x}^2m_{3y}^2] \quad (9)$$

$$= [m_{1x}m_{2x}m_{3x}], \quad (10)$$

where we dropped the λ to simplify the notation. Equations (6)–(10) imply that

$$E(\hat{D}) = m_{1x}m_{2x}m_{3x} = 1,$$

in clear contradiction with the quantum mechanical prediction given by (5), thus showing that noncontextual hidden-variables are incompatible with the quantum mechanical predictions. Hence, quantum mechanics is not compatible with the completeness requirements of EPR. We must stress at this point that the proof of the GHZ theorem relies on the fact that, in (10), all three terms on the left-hand side of the equation have the same hidden variable. However, if we allow the hidden variable to be contextual, i.e., for its value to depend on the experimental setup given by the observable operators, then no contradiction can be derived, as can be shown by proving that a nonmonotonic upper probability distribution exists that is compatible with the quantum mechanical expectations [6].

The main characteristic of GHZ's proof, however, has a major problem. How can we verify experimentally predictions based on correlation-one statements, since experimentally one cannot obtain events perfectly correlated? This problem was also present in Bell's original paper, where he considered cases where the correlations were one. To "avoid Bell's experimentally unrealistic restrictions", Clauser, Horne, Shimony and Holt [7] derived a new set of inequalities that would take into account imperfections in the measurement process. However, Bell's inequalities are quite different from the GHZ case, where in the latter it is *necessary* to have experimentally unrealistic perfect correlations to justify the proof leading to a contradiction. This can be seen from the following theorem (a simplified version of the theorem found in [8]), where the previous proof is reduced to pure probabilistic arguments.

Theorem 1 Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1, \mathbf{Y}_2$ and $\mathbf{Y}_3$ be six ± 1 random variables, representing the outcome of spin measurements in the x and y directions for particles 1, 2, and 3, and let

$$(i) \ E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) = 1,$$

$$(ii) \ E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = -1.$$

Then (i)–(ii) imply a contradiction.

Proof: First, we should note that

$$E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) = E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3), \quad (11)$$

since $\mathbf{Y}_i^2 = 1$. To simplify the notation, let us call $\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3 = \mathbf{A}$, $\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3 = \mathbf{B}$, $\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3 = \mathbf{C}$, and $\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3 = \mathbf{D}$. Then, (11) becomes

$$E(\mathbf{ABC}) = E(\mathbf{D}).$$

By definition

$$E(\mathbf{A}) = P(a) - P(\bar{a}), \quad (12)$$

where we used the notation a meaning $\mathbf{A} = 1$, $\bar{a}$ meaning $\mathbf{A} = -1$, and so on. But since $0 \leq P(a)$, $P(\bar{a}) \leq 1$, it follows at once from (ii) that

$$P(a) = 1 \quad (13)$$

and similarly

$$P(b) = P(c) = 1. \quad (14)$$

We also have that

$$P(\bar{d}) = 1.$$

Using again the definition of expectation and the inequalities $P(\bar{a}bc) \leq P(\bar{a}) = 0$, etc., we have

$$E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3) = P(abc) + P(\bar{a}\bar{b}c) + P(a\bar{b}\bar{c}) + P(\bar{a}b\bar{c}) \quad (15)$$

$$= P(abc) - [P(\bar{a}bc) + P(a\bar{b}c) + P(ab\bar{c}) + P(\bar{a}\bar{b}\bar{c})] = 1, \quad (16)$$

from (13) and (14), since all but the first term on the right is 0, and thus by conservation of probability $P(abc) = 1$. But (16) contradicts (ii). $\diamond$

In a previous paper [9] we analyzed the Innsbruck three-particle GHZ experiment [10], where a GHZ state was generated, and showed that their experiment was not compatible with a noncontextual hidden-variable theory. To do this, we showed that a set of inequalities for three-particle correlations were necessary and sufficient for the existence of a noncontextual hidden-variable. However, our proof did not use the full set of six detection random variables $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$, but instead used random variables defined as products of the $\mathbf{X}_i$'s and $\mathbf{Y}_i$'s, in a way similar to that used in the proof of Theorem 1. As a consequence, the set of inequalities presented in [9] is not symmetric in all random variables, as are Bell's inequalities. The present paper can be considered as a continuation of our early work [9], as we obtain conditions that, if verified experimentally, guarantee the nonexistence of a joint probability distribution for the particle observables $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$, and hence the nonexistence of a hidden-variable [11]. The main difference from the inequalities obtained here and the ones from [9] is that, in the form presented below in Theorem 2, the inequalities assume a completely symmetric form. Also, since the publication of [9] new data appeared from the group in Innsbruck [12], and in Section 3 we refine the analysis made previously in the light of these new data, concluding once again that no noncontextual hidden variables are compatible with the outcomes of the experiment.

2 Bell-like inequalities for the GHZ state

In this Section we want to derive inequalities that guarantee the nonexistence of hidden-variables for the observables used in GHZ's proof. Before we derive those inequalities, it is important to note that if we could measure all the random variables $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$ of Theorem 1 simultaneously, we would have a joint probability distribution. The existence of a joint probability distribution is a necessary and sufficient condition for the existence of a noncontextual hidden variable [11]. Hence, if the quantum mechanical GHZ correlations are obtained, then no such hidden variable exists. However, Theorem 1 still involves probability-one statements. On the other hand, the quantum mechanical correlations present in the GHZ state are so strong that even if we allow for experimental errors, the non existence of a joint distribution can still be verified, as we show in the following theorem, which, as we said above, extends the results in [9].

Theorem 2 Let $\mathbf{X}_i$ and $\mathbf{Y}_i$, $1 \leq i \leq 3$, be six ± 1 random variables. Then, there exists a joint probability distribution for all six random variables if and only if the following inequalities are satisfied:

$$-2 \leq E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{X}_3) - E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3) \leq 2,$$

$$-2 \leq -E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{X}_3) + E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3) \leq 2,$$

$$-2 \leq E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{Y}_3) - E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{X}_3) + E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3) \leq 2,$$

$$-2 \leq E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{Y}_3) - E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{X}_3) + E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3) \leq 2.$$

Proof: The argument is similar to the one found in [9]. To simplify, we use a notation where $x_1y_2y_3$ means $\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3 = 1$, $\overline{x_1y_2y_3}$ means $\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3 = -1$. We prove first that the existence of a joint probability distribution implies the four inequalities. Then, we have by an elementary probability computation that

$$P(x_1y_2y_3) = P(x_1y_2y_3, y_1x_2y_3, y_1y_2x_3) + P(x_1y_2y_3, \overline{y_1x_2y_3}, y_1y_2x_3) \\ + P(x_1y_2y_3, y_1x_2y_3, \overline{y_1y_2x_3}) + P(x_1y_2y_3, \overline{y_1x_2y_3}, \overline{y_1y_2x_3})$$

and

$$P(\overline{x_1y_2y_3}) = P(\overline{x_1y_2y_3}, y_1x_2y_3, y_1y_2x_3) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, y_1y_2x_3) \\ + P(\overline{x_1y_2y_3}, y_1x_2y_3, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}),$$

with similar equations for $\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3$ and $\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3$. But

$$\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3 = (\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3)(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3),$$

and so we have that

$$P(x_1x_2x_3) = P(x_1y_2y_3, y_1x_2y_3, y_1y_2x_3) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, y_1y_2x_3) \\ + P(x_1y_2y_3, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, y_1x_2y_3, \overline{y_1y_2x_3})$$

and

$$P(\overline{x_1x_2x_3}) = P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, y_1x_2y_3, y_1y_2x_3) \\ + P(x_1y_2y_3, \overline{y_1x_2y_3}, y_1y_2x_3) + P(x_1y_2y_3, y_1x_2y_3, \overline{y_1y_2x_3}).$$

A straightforward computation shows that

$$F = 2[P(x_1y_2y_3, y_1x_2y_3, y_1y_2x_3) + P(\overline{x_1y_2y_3}, y_1x_2y_3, y_1y_2x_3) \\ + P(x_1y_2y_3, \overline{y_1x_2y_3}, y_1y_2x_3) + P(x_1y_2y_3, y_1x_2y_3, \overline{y_1y_2x_3})] \\ - 2[P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, y_1y_2x_3) \\ + P(\overline{x_1y_2y_3}, y_1x_2y_3, \overline{y_1y_2x_3}) + P(x_1y_2y_3, \overline{y_1x_2y_3}, \overline{y_1y_2x_3})],$$

where F is defined by

$$F = E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) - E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3).$$

Since all probabilities are non-negative and sum to ≤ 1 , we infer the first inequality at once. The derivation of the other inequalities is similar.

Now for the sufficiency part. First, we assume the symmetric case where

$$E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) = 2p - 1, \quad (17)$$

and

$$E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = -(2p - 1). \quad (18)$$

Then, the first inequality yields

$$\frac{1}{4} \leq p \leq \frac{3}{4}, \quad (19)$$

while the other ones yield

$$0 \leq p \leq 1. \quad (20)$$

Since $\mathbf{X}_i$ and $\mathbf{Y}_i$ are ± 1 random variables, p has to belong to the interval $[0, 1]$, and inequality (20) doesn't add anything new. We will prove the existence of a joint probability distribution for this symmetric case by showing that, given any p , $\frac{1}{4} \leq p \leq \frac{3}{4}$, we can assign values to the atoms that have the proper marginal distributions.

The probability space for $\mathbf{X}_i$ and $\mathbf{Y}_i$ has 64 atoms. It is difficult to handle a problem of this size, so we will assume some further symmetries to reduce the number of atoms. We will first introduce the following notation: if a group of symbols is between square brackets, all the possible permutations of the barred symbol will be considered. For example, we assume $a_5 = P([\bar{x}_1 x_2 x_3] y_1 y_2 y_3)$, meaning that $P(\bar{x}_1 x_2 x_3 y_1 y_2 y_3) = a_5$, $P(x_1 \bar{x}_2 x_3 y_1 y_2 y_3) = a_5$, and $P(x_1 x_2 \bar{x}_3 y_1 y_2 y_3) = a_5$. Then, the number of atoms in the problem is reduced to the following 16: $a_1 = P(x_1 x_2 x_3 y_1 y_2 y_3)$, $a_2 = P(x_1 x_2 x_3 \bar{y}_1 \bar{y}_2 \bar{y}_3)$, $a_3 = P(x_1 x_2 x_3 [\bar{y}_1 y_2 y_3])$, $a_4 = P(x_1 x_2 x_3 [\bar{y}_1 \bar{y}_2 y_3])$, $a_5 = P([\bar{x}_1 x_2 x_3] y_1 y_2 y_3)$, $a_6 = P([\bar{x}_1 x_2 x_3] \bar{y}_1 \bar{y}_2 \bar{y}_3)$, $a_7 = P([\bar{x}_1 x_2 x_3] [\bar{y}_1 y_2 y_3])$, $a_8 = P([\bar{x}_1 x_2 x_3] [\bar{y}_1 \bar{y}_2 y_3])$, $a_9 = P([\bar{x}_1 \bar{x}_2 x_3] [\bar{y}_1 y_2 y_3])$, $a_{10} = P([\bar{x}_1 \bar{x}_2 x_3] [\bar{y}_1 \bar{y}_2 y_3])$, $a_{11} = P([\bar{x}_1 \bar{x}_2 x_3] y_1 y_2 y_3)$, $a_{12} = P([\bar{x}_1 \bar{x}_2 x_3] \bar{y}_1 \bar{y}_2 \bar{y}_3)$, $a_{13} = P(\bar{x}_1 \bar{x}_2 \bar{x}_3 [\bar{y}_1 y_2 y_3])$, $a_{14} = P(\bar{x}_1 \bar{x}_2 \bar{x}_3 [\bar{y}_1 \bar{y}_2 y_3])$, $a_{15} = P(\bar{x}_1 \bar{x}_2 \bar{x}_3 y_1 y_2 y_3)$, $a_{16} = P(\bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{y}_1 \bar{y}_2 \bar{y}_3)$.

These new added symmetries reduce the problem from 64 to 16 variables. The atoms have to satisfy various sets of equations. The first set comes just from the requirement that $E(\mathbf{X}_i) = E(\mathbf{Y}_i) = 0$, for $i = 1, 2, 3$, but two of the six equations are redundant, and so we are left with the following four.

$$\begin{aligned} a_1 + a_2 + 3a_3 + 3a_4 + a_5 + a_6 + 3a_7 + 3a_8 - 3a_9 \\ - 3a_{10} - a_{11} - a_{12} - 3a_{13} - 3a_{14} - a_{15} - a_{16} &= 0, \end{aligned} \quad (21)$$

$$\begin{aligned} a_1 - a_2 + a_3 - a_4 + 3a_5 - 3a_6 + 3a_7 - 3a_8 + 3a_9 \\ - 3a_{10} + 3a_{11} - 3a_{12} + a_{13} - a_{14} + a_{15} - a_{16} &= 0, \end{aligned} \quad (22)$$

$$\begin{aligned} a_1 - a_2 + a_3 - a_4 + 3a_5 - 3a_6 + 3a_7 - 3a_8 + 3a_9 \\ - 3a_{10} + 3a_{11} - 3a_{12} - a_{13} + a_{14} + a_{15} - a_{16} &= 0, \end{aligned} \quad (23)$$

$$\begin{aligned} a_1 + a_2 + 3a_3 + 3a_4 - a_5 - a_6 - 3a_7 - 3a_8 + 3a_9 \\ + 3a_{10} + a_{11} + a_{12} - 3a_{13} - 3a_{14} - a_{15} - a_{16} &= 0, \end{aligned} \quad (24)$$

where (21) comes from $E(\mathbf{X}_1) = 0$, (22) from $E(\mathbf{X}_2) = 0$, (23) from $E(\mathbf{Y}_1) = 0$, and (24) from $E(\mathbf{Y}_2) = 0$. The triple expectations also imply

$$\begin{aligned} a_1 + a_2 + 3a_3 + 3a_4 - 3a_5 - 3a_6 - 9a_7 - 9a_8 + 9a_9 \\ + 9a_{10} + 3a_{11} + 3a_{12} - 3a_{13} - 3a_{14} - a_{15} - a_{16} &= -2p + 1, \end{aligned} \quad (25)$$

$$\begin{aligned} a_1 + a_2 - a_4 - a_5 + a_6 - a_7 - a_8 + a_9 \\ + a_{10} - a_{11} - a_{12} + a_{13} + a_{14} - a_{15} - a_{16} &= 2p - 1, \end{aligned} \quad (26)$$

and

$$\begin{aligned} a_1 - a_2 - 3a_3 + 3a_4 + 3a_5 - 3a_6 - 9a_7 + 9a_8 - 9a_9 \\ + 9a_{10} + 3a_{11} - 3a_{12} - 3a_{13} + 3a_{14} + a_{15} - a_{16} &= 2p - 1. \end{aligned} \quad (27)$$

Finally, the probabilities of all atoms have to sum to one, yielding the last equation

$$\begin{aligned} a_1 + a_2 + 3a_3 + 3a_4 + 3a_5 + 3a_6 + 9a_7 + 9a_8 + 9a_9 \\ + 9a_{10} + 3a_{11} + 3a_{12} + 3a_{13} + 3a_{14} + a_{15} + a_{16} &= 1. \end{aligned} \quad (28)$$

Even with the symmetries reducing the problem to 16 variables, we still have an infinite number of solutions that satisfy Eqns. (21)–(28). Since it is very hard to exhibit a general solution for (21)–(28) and the constraints $0 \leq a_i \leq 1$, $i = 1 \dots 16$, we will just show that particular solutions exist for an arbitrary p satisfying the inequality (19). To do so, we will divide the problem into two parts: one where we will exhibit an explicit solution for the atoms $a_1, \dots, a_{16}$ that form a proper probability distribution for $p \in [\frac{1}{4}, \frac{1}{2}]$, and another explicit solution for $p \in [\frac{1}{2}, \frac{3}{4}]$.

It is easy to verify that, given an arbitrary p in $[\frac{1}{4}, \frac{1}{2}]$, the following set of values constitute a solution of Eqns. (21)–(28): $a_1 = 0, a_2 = -\frac{1}{2} + 2p, a_3 = \frac{1}{4} - \frac{1}{2}p, a_4 = 0, a_5 = 0, a_6 = 0, a_7 = 0, a_8 = 0, a_9 = 0, a_{10} = 0, a_{11} = 0, a_{12} = \frac{1}{4} - \frac{1}{2}p, a_{13} = 0, a_{14} = 0, a_{15} = p, a_{16} = 0$. For p in $[\frac{1}{2}, \frac{3}{4}]$ the following set of values constitute a solution of Eqns. (21)–(28): $a_1 = -\frac{1}{8} + \frac{1}{2}p, a_2 = 0, a_3 = \frac{3}{8} - \frac{1}{2}p, a_4 = 0, a_5 = -\frac{5}{24} + \frac{1}{3}p, a_6 = -\frac{1}{24} + \frac{1}{6}p, a_7 = 0, a_8 = 0, a_9 = 0, a_{10} = 0, a_{11} = 0, a_{12} = 0, a_{13} = 0, a_{14} = \frac{1}{8}, a_{15} = \frac{3}{8} - \frac{1}{2}p, a_{16} = 0$. So, for p satisfying the inequality $\frac{1}{4} \leq p \leq \frac{3}{4}$ we can always construct a probability distribution for the atoms consistent with the marginals, and this concludes the proof. $\diamond$

We note that the form of the inequalities of Theorem 2 is actually that of the Clauser et al. inequalities [7] for the Bell case, when the Bell binary correlations are replaced by the GHZ triple correlations. The inequalities from Theorem 2 immediately yield the following.

Corollary Let $\mathbf{X}_i$ and $\mathbf{Y}_i$, $1 \leq i \leq 3$, be six ± 1 random variables, and let

$$(i) \ E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{Y}_3) = E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{Y}_3) = E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{X}_3) = 1 - \varepsilon,$$

$$(ii) \ E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3) = -1 + \varepsilon,$$

$\varepsilon \in [0, 1]$. Then there cannot exist a joint probability distribution of $\mathbf{X}_i$ and $\mathbf{Y}_i$, $1 \leq i \leq 3$, satisfying (i) and (ii) if $\varepsilon < \frac{1}{2}$.

Proof. If a joint probability exists, then

$$-2 \leq E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{X}_3) - E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3) \leq 2.$$

But

$$E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{Y}_3) + E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{X}_3) - E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3) = 4 - 4\varepsilon,$$

and the inequality is satisfied only if $\varepsilon \geq \frac{1}{2}$. Hence, if $\varepsilon < \frac{1}{2}$ no joint probability exists. $\diamond$

In the Corollary, ε may represent, for example, a deviation, due to experimental errors, from the predicted quantum mechanical correlations. So, we see that to prove the nonexistence of a joint probability distribution for the GHZ experiment, we do not need to have perfect measurements and 1 or -1 correlations. In fact, from the inequality obtained above, it is clear that any experiment that satisfies the strong symmetry of the Corollary and obtains a correlation for the triples stronger than 0.5 (and -0.5 for one of them) cannot have a joint probability distribution. It is worth mentioning at this point that the inequalities derived in Theorem 2 have a completely different origin than do Bell's inequalities. The inequalities of Theorem 2 are not satisfied by a particular model, but they just accommodate the theoretical conditions in *GHZ* to possible experimental deviations. Also, Theorem 2 does not rely on any “enhancement” hypothesis to reach its conclusion. Thus, with this reformulation of the *GHZ* theorem it is possible to use strong, yet imperfect, experimental correlations to prove that a hidden-variable theory is incompatible with the experimental results.

To complete this Section, we will now present a result that is similar to one obtained by Garg and Mermin for the case of Bell's inequalities [13]. Garg and Mermin constructed a set of eight ± 1 random variables that satisfy Bell's inequalities, but do not have a joint probability distribution. For our example, it is sufficient to increase the number of random variables from six to seven.

Theorem 3 Let $\mathbf{X}_i$, $\mathbf{Y}_i$, and $\mathbf{Y}'_1$, $1 \leq i \leq 3$, be seven ± 1 random variables with expectations equal to zero. The satisfaction of the inequalities of Theorem 2 by the subsets of random variables $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3\}$ and $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}'_1, \mathbf{Y}_2, \mathbf{Y}_3\}$ does not guarantee the existence of a joint probability distribution for the full set of random variables.

Proof. We prove Theorem 3 by constructing an example of seven variables not having a joint probability distribution. Say we have the following expectations, all satisfying the inequalities of Theorem 2:

$$E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{X}_3) = E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{X}_3) = E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{Y}_3) = E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{Y}_3) = -1, \quad (29)$$

$$E(\mathbf{Y}'_1 \mathbf{X}_2 \mathbf{X}_3) = E(\mathbf{Y}'_1 \mathbf{Y}_2 \mathbf{Y}_3) = 0, \ E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{X}_3) = E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{Y}_3) = -1. \quad (30)$$

Since $E(\mathbf{Y}'_1 \mathbf{X}_2 \mathbf{X}_3) = E(\mathbf{Y}'_1 \mathbf{Y}_2 \mathbf{Y}_3) = 0$, a joint probability for the six variables of (30) exists with necessarily the correlations

$$E(\mathbf{Y}'_1 \mathbf{Y}_2) = 1, \quad E(\mathbf{Y}'_1 \mathbf{X}_2) = 1, \quad E(\mathbf{Y}'_1 \mathbf{Y}_3) = 1, \quad E(\mathbf{Y}'_1 \mathbf{X}_3) = -1.$$

From the correlations fixed above, we can obtain immediately that

$$E(\mathbf{X}_2 \mathbf{Y}_2) = 1, \quad E(\mathbf{X}_3 \mathbf{Y}_3) = -1.$$

But, we have also that

$$E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{Y}_3) = E((\mathbf{Y}_1 \mathbf{X}_2 \mathbf{X}_3)(\mathbf{X}_2 \mathbf{Y}_2)(\mathbf{X}_3 \mathbf{Y}_3)) = -E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{X}_3),$$

thus showing that the assumption of a joint probability distribution leads to a contradiction with (29). $\diamond$

The counterexample to the existence of a joint probability distribution just constructed, and also the one constructed by Garg and Mermin for Bell's inequalities, simply illustrate a well-known feature of probability theory that has nothing to do with quantum mechanics. The feature is this. Given marginal probability distributions for pairs, triples, or, in general, n -tuples, $n = 1, 2, \dots$, of random variables, there will not exist a joint probability distribution for m -tuples of random variables, $m > n$, without making special assumptions. Of course, distributions satisfying special assumptions are of great importance, e.g., the Gaussian or normal distribution of n random variables for which only means, variances and covariances are needed to fix the distribution uniquely. Consequently, for a Gaussian, all higher moments are already fixed by the first and second moments.

It is also to be emphasized that giving inequalities for larger numbers of random variables, either in the Bell or GHZ case, necessary and sufficient for the existence of a joint probability distribution is difficult, and very likely intractable for arbitrary n . For a sense of the difficulties in the Bell case, see Suppes and Zanotti [11].

3 Analysis of the Innsbruck Experiment

In this Section we will study the expected correlations for a GHZ-like experiment when detector inefficiencies are taken into account. At the end of this Section, our results will be compared to the most recent experimental results of the Innsbruck group [10, 12], as well as to our previous work [9]. Before we proceed, we need to lay down the experimental setup that we will consider. A schematic of the Innsbruck experiment is shown in Figure 1. In this experiment an UV laser pulse is sent into a nonlinear crystal, generating correlated pairs of photons, that can be detected in four detectors T , D_1 , D_2 , and D_3 . In the Figure, "Pol." are polarizers, "BS" is a 50-50 beam splitter, and $\lambda/2$ is a half-wave plate. There is a small, yet nonzero, probability that two pairs of photons are generated simultaneously, where by simultaneous we mean within a window of observation Δt . It can be shown theoretically that, for the case of two-pair production, when one of the photons is detected in the trigger detector T , the other three photons detected in D_1 , D_2 , and D_3 are in a GHZ entangled state. In other words, if we conditionalize the experimental data to a detection at T , then D_1 , D_2 , and D_3 should show a GHZ correlation. There are, however, several different sources of inefficiencies that we have to take into account in a GHZ-like experiment: the detection efficiency, the dark-count rate, and the misfiring probability, just to name a few, but important, examples. The detection efficiency, represented in this paper by a parameter $d \in [0, 1]$, is the probability of a photon being detected by the detector. According to the definition of d , if a photon arrives at a detector D , the probability of this detector firing is d . The detection efficiency is an important factor in analyzing real experiments, as usually d is lower than 0.7 for photons in the frequency of the visible spectrum. The dark-count rate, denoted γ , is another important factor in experiments dealing with correlations, as it represents the frequency of firings that a detector registers if *no* photon reaches the detector. The misfiring probability is related to inefficiencies of the polarizers that result in confusing a $+$ -polarized photon with a $-$ -polarized photon, for example.

In this Section we give an analysis of what happens to the GHZ correlations in a setup like Figure 1 when the detectors have efficiency $d \in [0, 1]$ and a dark-count rate of probability γ . We will assume that the misfiring probabilities are negligible, as the polarizers used in the wavelengths in question are very efficient.

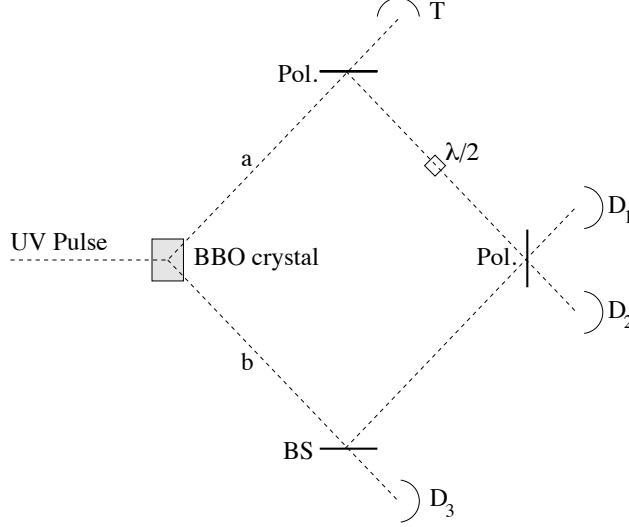


Figure 1: Schematics of the GHZ experiment of Bouwmeester et al.

In our analysis we will also assume that whenever a double pair is produced, this double pair has the expected GHZ correlation for D_1 , D_2 , and D_3 if T correctly detects a photon. We will consider negligible the probability of no photon being generated and having a fourfold coincidence registered, as this would require a dark-count rate much bigger than found in modern photon detectors. Also, triple-pair creations are going to be neglected because they are of extremely low probability compared to the other events. Our analysis in this Section follows closely our previous one in [9].

The problem can be naturally divided in two parts: i) a pair of photons is generated at the parametric down converter; ii) two pair of photons are generated. We will denote by p_1p_2 the generation of two photons, and by $p_1...p_4$ that generation of four photons. We will assume that $P(p_1...p_4) + P(p_1p_2) = 1$, as other events are of negligible probability when we condition to a fourfold detection ($T, D_1, ..., D_3$).

Let us start with the photon case, p_1p_2 . The two photons p_1p_2 can reach, as one goes through channel a and the other through channel b (see Figure 1), any of the following combinations of detectors: $TD_1, TD_2, TD_3, D_1D_1, D_1D_2, D_1D_3, D_2D_2, D_2D_3, (D_3D_3$ and TT are not allowed, as both photons cannot go through the same channel) where D_iD_i represent when both photons reach the same detector. Events are selected only if the four detectors fire, as this is a requirement for GHZ correlations. If the detectors were ideal, two photon states would not be included in the experimental data table. However, since detectors sometimes fire when a photon is not present due to dark-count rates, we have to take this into account. We take as our set of random variables T, D_1, D_2, D_3 which take values 1 or 0. To simplify the notation, we will use t, d_1, d_2 , and d_3 to represent a measurement at the detectors (i.e. 1), and $\bar{t}, \bar{d}_1, \bar{d}_2, \bar{d}_3$ to represent 0. We want to compute

$$P(td_1d_2d_3 | p_1p_2), \quad (31)$$

the probability that all detectors T, D_1, D_2, D_3 fire simultaneously, with the condition that only a pair p_1p_2 of photons has been produced at the BBO crystal. Let us first compute the case where the two photons arrive at detectors T and D_3 . Since the efficiency of the detectors is d , the probability that both detectors detect the photons is d^2 , the probability that only one detects is $2d(1-d)$ and the probability that neither of them detects is $(1-d)^2$. Then, the probability that all four detectors fire is

$$P(td_1d_2d_3 | p_1p_2 = TD_3) = \gamma^2 (d + \gamma(1-d))^2, \quad (32)$$

where $p_1p_2 = TD_3$ represents the simultaneous (i.e. within a measurement window) arrival of the photons at the trigger T and at D_3 , and where γ is the probability of a dark count, as defined above. Similar computations can be carried out for $p_1p_2 = TD_1, TD_2, D_1D_3, D_1D_2, D_2D_3$. For $p_1p_2 = D_iD_i$ the computation of $P(td_1d_2d_3 | p_1p_2 = D_iD_i)$ is different. We assume that the probability that exactly one of the photons is detected at D_i is $d(1-d)$ and the probability that neither of them is detected is $(1-d)^2$. Then, it is clear

that

$$P(td_1d_2d_3 | p_1p_2 = D_iD_i) = d(1-d)\gamma^3 + (1-d)^2\gamma^4.$$

Then, we have that

$$P(td_1d_2d_3 | p_1p_2) = 6\gamma^2(d + \gamma(1-d))^2 + 2\gamma^3(1-d)(d + \gamma).$$

At this point we should stress that, contrary to true GHZ detections, events involving $P(td_1d_2d_3 | p_1p_2)$ are expected to have no spin correlation, as the fourfold coincidence is due mainly to dark counts, i.e. to random events.

We now turn our attention to the case where four photons, $p_1...p_4$, are present. The probability that all the four photons are detected is d^4 , that three of the four are detected is $4d^3(1-d)$, that two are detected is $6d^2(1-d)^2$, that only one is detected is $4d(1-d)^3$, and that none are detected is $(1-d)^4$. If all four are detected, we have a true GHZ state detected, and the correlation is the expected one for the GHZ case. However, one can again have four photons detected due to dark-count rates. To simplify once again the notation, we will write $p_1...p_4 = GHZ$ to represent having the four photons as a GHZ state, and $p_1...p_4 = \overline{GHZ}$ as having the four photons as a non-GHZ state. We then have

$$P(td_1d_2d_3 | p_1...p_4 = GHZ) = d^4 + \gamma(1-d)d^3 \quad (33)$$

and

$$P(td_1d_2d_3 | p_1...p_4 = \overline{GHZ}) = 3\gamma d^3(1-d) + 6\gamma^2 d^2(1-d)^2 + 4\gamma^3 d(1-d)^3 + \gamma^4(1-d)^4. \quad (34)$$

The last term in (33) comes from the unique role of the trigger T , that needs to detect a photon but whose photon need not have a GHZ correlation.

How do the non-GHZ detections change the GHZ expectations? What is measured in the laboratory is the quantity $E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | td_1d_2d_3)$, where $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ are random variables with values ± 1 , representing the spin measurement at D_1 , D_2 and D_3 respectively. We can write it as

$$E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | td_1d_2d_3) = \frac{E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | td_1d_2d_3 \& GHZ) P(GHZ)}{P(GHZ) + P(\overline{GHZ})}.$$

since for non-GHZ states we expect a correlation zero for the term

$$\frac{E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | td_1d_2d_3 \& \overline{GHZ}) P(\overline{GHZ})}{P(GHZ) + P(\overline{GHZ})}.$$

Neglecting terms of higher orders in γ , using $\gamma \ll d$, and $P(p_1p_2) \gg P(p_1...p_4)$, we obtain, from $P(\overline{GHZ}) = 6P(p_1p_2)\gamma^2d^2 + 3P(p_1...p_4)\gamma d^3(1-d)$ and $P(GHZ) = P(p_1...p_4)[d^4 + \gamma(1-d)d^3]$, that

$$E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | td_1d_2d_3) = \frac{E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | td_1d_2d_3 \& GHZ)}{\left[1 + 6\frac{P(p_1p_2)}{P(p_1...p_4)}\frac{\gamma^2}{d^2}\right]}. \quad (35)$$

This value is the corrected expression for the correlations if we have detector efficiency taken into account.

The product $\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3$ of the random variables can take only values $+1$ or -1 . Then, if their expectation is $E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | T)$ we have

$$P(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 = 1 | td_1d_2d_3) = \frac{1 + E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | td_1d_2d_3)}{2}.$$

The variance σ^2 for a random variable that assumes only 1 or -1 values is $4P(1)(1 - P(1))$. Hence, in our case we have as a variance

$$\sigma^2 = \left[1 - E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | td_1d_2d_3)^2\right].$$

We will estimate the values of γ and d to see how much $E(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 | td_1d_2d_3)$ would change due to experimental errors. For that purpose, we will use typical rates of detectors [14] for the frequency used at

the Innsbruck experiment, as well as their reported data [10]. First, modern detectors usually have $d \approx 0.5$ for the wavelengths used at Innsbruck. We assume a dark-count rate of about 3×10^2 counts/s. With a time window of coincidence measurement of 2×10^{-9} s, we then have that the probability of a dark count in this window is $\gamma = 6 \times 10^{-7}$. From [10] we use that the ratio $P(p_1 p_2)/P(p_1 \dots p_2)$ is on the order of 10^{10} . Substituting these numerical values in (35) we have $E(\mathbf{S}_1 \mathbf{S}_2 \mathbf{S}_3 | td_1 d_2 d_3) \cong 0.9$. From this expression it is clear that the change in correlation imposed by the dark-count rates is significant for the given parameters. However, it is also clear that the value of the correlation is quite sensitive to changes in the values of both γ and d .

We can now compare the values we obtained with the ones observed by Bouwmeester et al. for GHZ and $\overline{GHZ}$ states [12]. Using the notation from previous sections, they report that for the settings $\mathbf{Y}_1 \mathbf{X}_2 \mathbf{X}_3$, $\mathbf{X}_1 \mathbf{Y}_2 \mathbf{X}_3$, and $\mathbf{X}_1 \mathbf{X}_2 \mathbf{Y}_3$ a fraction of 0.85 ± 0.04 had the predicted quantum mechanical correlation, i.e.,

$$E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{X}_3) + E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{X}_3) + E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{Y}_3) = 2.55.$$

However, for the experimental setting $\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{Y}_3$ they report a correlation

$$E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{Y}_3) = -0.74.$$

If we use the inequalities from Theorem 2, we see that the first inequality is clearly violated, as

$$E(\mathbf{Y}_1 \mathbf{X}_2 \mathbf{X}_3) + E(\mathbf{X}_1 \mathbf{Y}_2 \mathbf{X}_3) + E(\mathbf{X}_1 \mathbf{X}_2 \mathbf{Y}_3) - E(\mathbf{Y}_1 \mathbf{Y}_2 \mathbf{Y}_3) = 3.29 > 2.$$

Using Theorem 2, this result strongly supports the nonexistence of a joint probability distribution.

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